NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

PREPARATORY QUIZ

Problem 1. Let f be a function which maps into some Euclidean space $\mathbb{R}^m$ and whose domain is an open subset Ω of some other Euclidean space $\mathbb{R}^n$.

(a) Define the PARTIAL DERIVATIVES of f at $x \in \Omega$ (if they exist).
(b) What does it mean for f to be GATEAUX DIFFERENTIABLE at $x \in \Omega$?
(c) What does it mean for f to be FRÉCHET DIFFERENTIABLE at $x \in \Omega$?
(d) What, if any, is the relationship between Gateaux and Fréchet differentiability?

Problem 2. Compute the LAPLACIAN of the function $f : \mathbb{R}^n \to \mathbb{R}$, where

(a) $f(x) \doteqdot \frac{1}{2}|x|^2$.
(b) $f(x) \doteqdot \mathrm{e}^{x \cdot e_1}$.
(c) f is the product of two smooth functions $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ (in terms of the Laplacians and gradients of g and h).
(c) f is the quotient of two smooth functions $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ (in terms of the Laplacians and gradients of g and h).

Problem 3. Let $\{u_i : \overline{\Omega} \to \mathbb{R}\}_{i \in \mathbb{N}}$ be a sequence of C^1 functions, where Ω is a bounded open subset of $\mathbb{R}^n$. Suppose that

$$\sup_{i \in \mathbb{N}} \max_{\overline{\Omega}} \left(|u| + |Du| \right) < \infty \, .$$

Show that some subsequence of $\{u_i : \overline{\Omega} \to \mathbb{R}\}_{i \in \mathbb{N}}$ converges uniformly to some limit $u_\infty : \overline{\Omega} \to \mathbb{R}$. *Hint: you should invoke the* ARZELÀ–ASCOLI THEOREM.

Problem 4. Suppose that a sequence of continuous functions $u_i : \Omega \underset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}$ converges uniformly to a limit $u_\infty : \Omega \to \mathbb{R}$. Prove (from first principles) that u_∞ is continuous.

Problem 5. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a HARMONIC FUNCTION. That is, u is continuous and

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u \, d\mathcal{L}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where $\mathcal{L}$ is the Lebesgue measure and $|B_r(x)| \doteqdot \mathcal{L}(B_r(x))$. Suppose that u is bounded. Prove that u is constant. *Hint: given any two points $x, y \in \mathbb{R}^n$, $|B_r(x) \setminus B_r(y)| = o(r^n)$ as $r \to \infty$.*

Problem 6. Find all solutions of the form $Du|_{(x,y)} = (F(x), G(y))$ to the PDE

$$-|Du| \operatorname{div} \left(\frac{Du}{|Du|} \right) = 1 \ \text{ in } \ \Omega \underset{\text{open}}{\subset} \mathbb{R}^2 \, .$$

Problem 7. Denote by B the unit ball in $\mathbb{R}^n$. Suppose that u is smooth on $\overline{B}$ and satisfies the LAPLACE EQUATION

$$-\Delta u = 0$$

in B. Show that the Dirichlet energy

$$E(u) \div \frac{1}{2}\int_B |Du|^2 d\mathcal{L}$$

of u cannot be decreased by "small" smooth perturbations of u in the interior of B. That is, given any smooth function v which is non-zero only in a subset U whose closure lies in B, there exists $\varepsilon_0 > 0$ such that

$$E(u + \varepsilon v) > E(u)$$

for all $\varepsilon < \varepsilon_0$. *Hint: the* DOMINATED CONVERGENCE THEOREM *ensures that $E(u + \varepsilon v)$ is smooth (since derivatives with respect to ε may be commuted with the integral), so you may approximate $E(u + \varepsilon)$ using* TAYLOR'S THEOREM. *You will also need the* DIVERGENCE THEOREM.

Find the solutions here