Quiz

1. Let Ω be an open bounded domain with regular boundary. Consider the following elliptic problem

$$\begin{aligned} -\nabla \cdot (\alpha \nabla u) + \gamma u &= f && \text{in } \Omega, \\ -\alpha \partial_n u + \sigma u &= \phi_R && \text{on } \partial\Omega, \end{aligned}$$

where $\alpha > 0$, $\gamma > 0$, $\sigma > 0$ are suitable constants, while $\phi_R \in L^2(\partial\Omega)$ and $f \in L^2(\Omega)$ are given functions. Study the existence and uniqueness of the weak solution u depending on the parameters α, γ, σ.

2. Let $a : V \times V \to \mathbb{R}$ a bilinear form bounded and coercive, and $F \in V'$, with V a Hilbert space. Let $\{V_h\}_h$ be a family of finite dimensional subspaces of V, and for each h consider $F_h \in V_h'$ and a bounded bilinear form $a_h : V_h \times V_h \to \mathbb{R}$. Also assume that there exists $\tilde{\alpha} > 0$, independent of h such that $a_h(v_h, v_h) \geq \tilde{\alpha}\|v_h\|_V^2$, for all $v_h \in V_h$.

   (a) Show that there exist unique $u \in V$ and $u_h \in V_h$ such that

$$a(u, v) = F(v) \quad \forall v \in V \qquad \text{and } a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h.$$

   (b) (*Optional*) Show that there exists $C > 0$, independent of h, such that

$$\|u - u_h\|_V \leq C \left(\inf_{v_h \in V_h} \left\{ \|u - v_h\|_V + \sup_{w_h \in V_h \setminus \mathbf{0}} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_V} \right\} + \sup_{w_h \in V_h \setminus \mathbf{0}} \frac{|F(w_h) - F_h(w_h)|}{\|w_h\|_V} \right).$$

3. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be an open bounded domain and consider the steady Stokes problem:

$$\begin{aligned} -\nu \triangle \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{f} \in [L^2(\Omega)]^d$ is an assigned forcing term and the viscosity $\nu > 0$ is assumed constant.

   (a) Is the pressure p defined uniquely? (check whether $p + c$, with c constant, is also a solution to (1)).

   (b) Show that if $\mathbf{u}$ and p are solutions of (1), then they are also solutions of the following weak problem: find $\mathbf{u} \in \boldsymbol{V}$, $p \in Q_0$ such that

$$\nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} - \int_\Omega p \, \nabla \cdot \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \boldsymbol{V}, \tag{2}$$

$$\int_\Omega q \, \nabla \cdot \mathbf{u} = 0 \qquad \forall q \in Q_0, \tag{3}$$

   where

$$\boldsymbol{V} = [H_0^1(\Omega)]^d, \qquad Q_0 = \left\{ p \in L^2(\Omega) : \int_\Omega p = 0 \right\},$$

   and we have denoted

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i=1}^d \int_\Omega \nabla u_i \cdot \nabla v_i = \sum_{i,j=1}^d \int_\Omega \partial_j u_i \, \partial_j v_i \, .$$

   (c) Let us now consider the space $\boldsymbol{V}_{\text{div}}^0 = \{\mathbf{v} \in \boldsymbol{V} : \nabla \cdot \mathbf{v} = 0\}$. Show that if $\mathbf{u}$ is the solution of (2)–(3), then $\mathbf{u}$ satisfies the problem

$$\text{find } \mathbf{u} \in \boldsymbol{V}_{\text{div}}^0 : \quad \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in \boldsymbol{V}_{\text{div}}^0 \, . \tag{4}$$

   (d) Prove that (4) is well-posed.

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