# AMSI Summer School: Stochastic Transport Modelling Quiz

Consider a single agent moving on an infinite one-dimensional uniform lattice. Suppose that the agent starts at the origin, $x = 0$, and moves a short distance $\delta > 0$ either left or right in a short time interval of duration $\tau > 0$. The motion is completely random, so the probability of moving left or right are 1/2. This simple random walk proceeds by repeating each random step $n$ times:

1. Write down an expression for the expected displacement of the agent after $n = 1$ time step.

2. Write down an expression for the expected squared displacement of the agent after $n = 1$ time step.

3. Suppose that $p(m,n)$ is the probability that the agent resides $m$ lattice sites after taking $n$ steps. Note that $m$ and $n$ are integers. With this information, provide a hand-drawn sketch of $p(m,1)$, $p(m,2)$, $p(m,3)$ and $p(m,4)$ on the interval $-5 \leq m \leq 5$.

4. Using your insight from the previous question convince your self that the probability that an agent will be at a distance $m\delta$ to the right of the origin after $n$ time steps, where $m$ and $n$ are even, is given by

$$p(m,n) = \left(\frac{1}{2}\right)^n \binom{n}{[n-m]/2} = \frac{n!}{2^n((n+m)/2)!((n-m)/2)!} \quad (1)$$

5. Use a computer to plot $p(m,n)$ for large $n$ to visualise this discrete distribution. Take care with the range of $m$ over which you plot the distributions. What do you notice?

6. For large $n$ this distribution approaches a normal distribution, so after a large amount of time $t = n\tau$, the location $x = m\delta$ of the agent is normally distributed with zero mean and variance $\delta^2 t/\tau$. Constructing a limit $\delta \to 0$ and $\tau \to 0$ with care such that $\delta^2/\tau = 2D$, where $D$ is a constant called the diffusion constant, gives a continuous probability density function for the location of the agent at time $t$,

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right) \quad (2)$$

   Use a computer to make several plots of the discrete distribution $p(m,n)$ and the corresponding continuous distribution $p(x,t)$ and compare the distributions? Hint: be careful to ensure you are making a valid comparison by comparing $p(m,n)$ and $2 \times p(x,t)$ and think about why this is necessary.

7. Using your explorations can you identify and describe situations where one approach makes more sense than the other?

1. $\frac{1}{2} \times \delta + \frac{1}{2} \times -\delta = 0.$

2. $\frac{1}{2} \times (\delta)^2 + \frac{1}{2} \times (-\delta)^2 = \delta^2.$

3. See hand-drawn sketch attached.

4. Continuing with the pattern developed in the hand-drawn sketch we see that after an even number of steps that the agent can only be an even number of steps away from the origin, or after an off number of steps the agent can only be an odd number of steps away. Continuing with this pattern, after $n$ steps we have

$$p(m,n) = \left(\frac{1}{2}\right)^n \binom{n}{[n-m]/2}, \tag{3}$$

provided that $m$ and $n$ are even.

5. See code and plots. Plotting $p(m,n)$ as $n \gg 1$ we see that the discrete distribution becomes increasingly smooth by visual inspection.

6. See code and plots. Superimposing plots of $p(m,n)$ and $2p(x,t)$ we see computational evidence that $p(m,n) \to 2p(x,t)$ as $n \to \infty$.

7. Comparing $p(m,n)$ and $2p(x,t)$ reveals the approximate nature of the continuous distribution. In the discrete distribution we always have $p(m,n) = 0$ for all locations and times satisfying $|m| > n$ since it is physically impossible for an agent to move away from the origin faster than one site per time step. In comparison, with the continuous distribution we have $p(x,t) > 0$ for all $-\infty < x < \infty$, no matter how small $t$ is! The continuous approximation is, in this sense completely unphysical. This observation is a good reminder that we are working with an approximation that is sometimes useful but can also be misleading.

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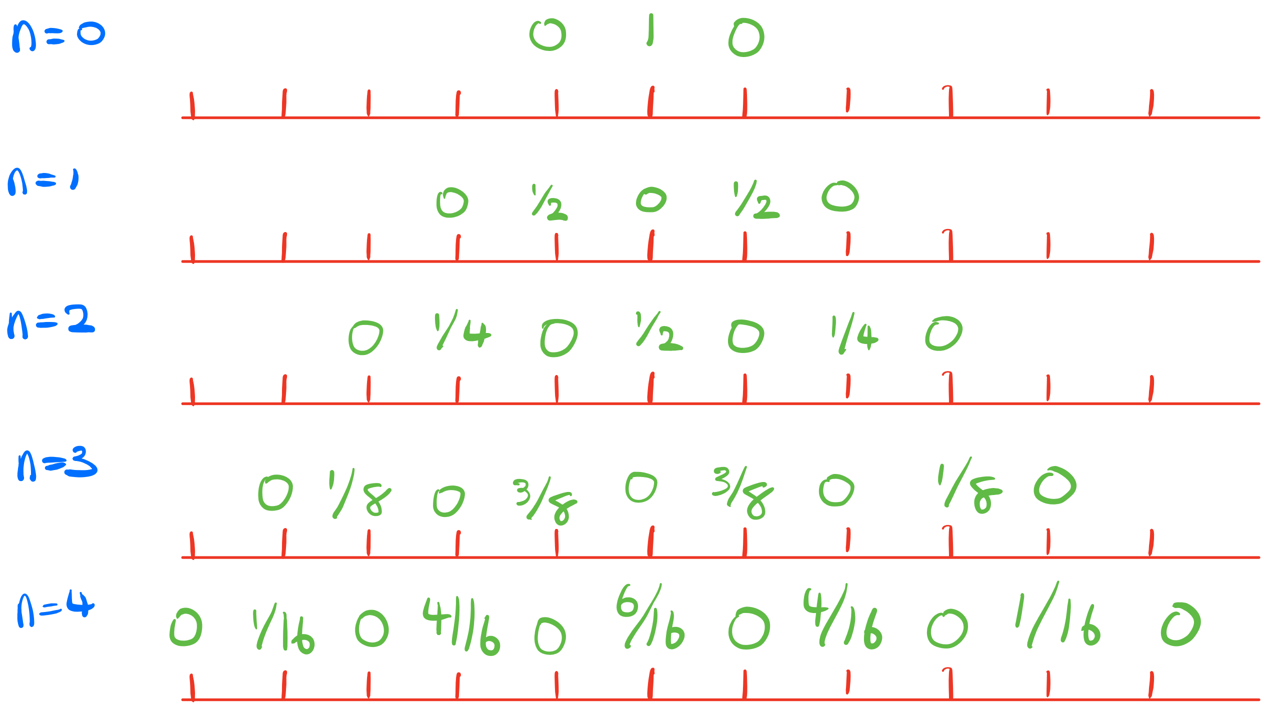
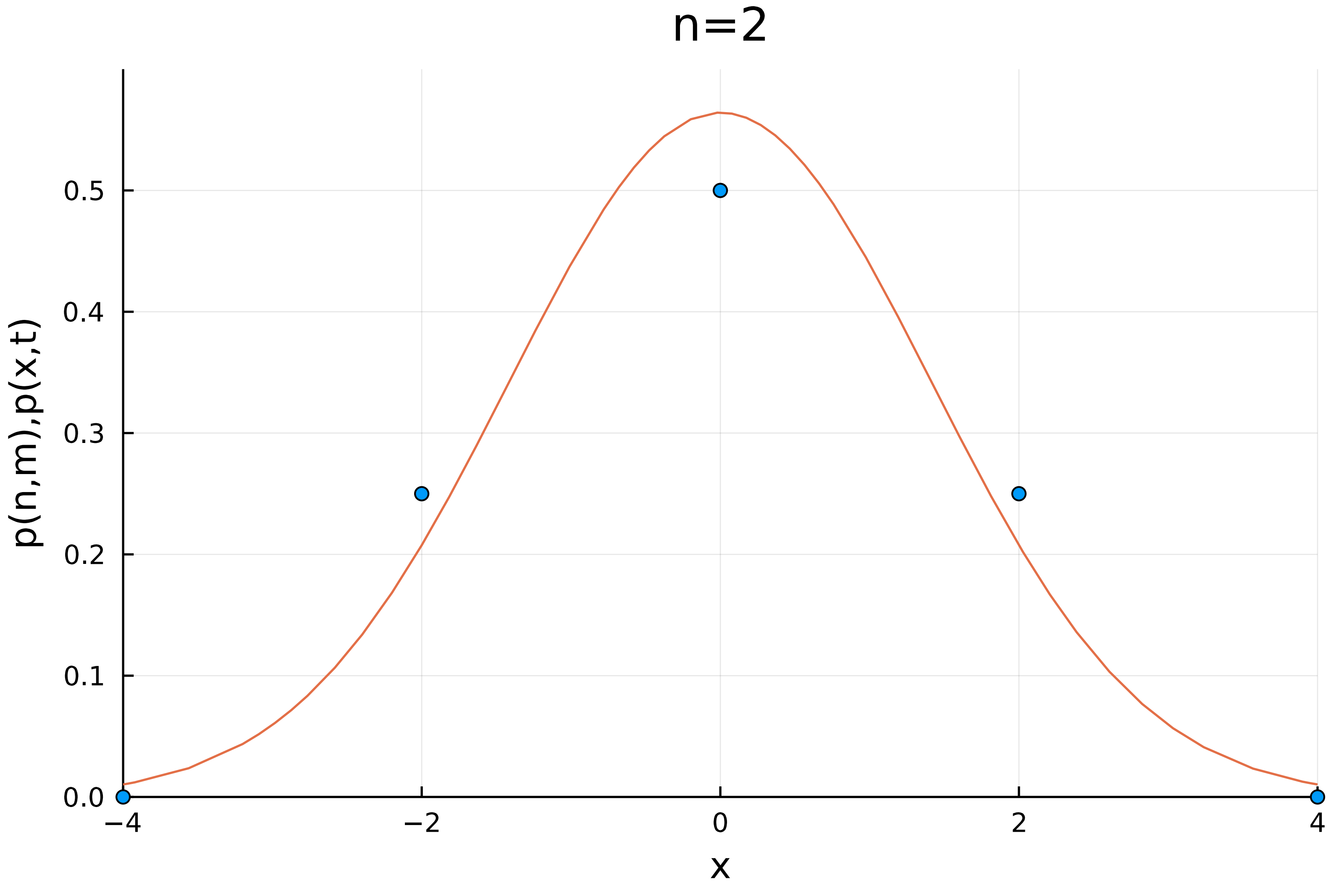
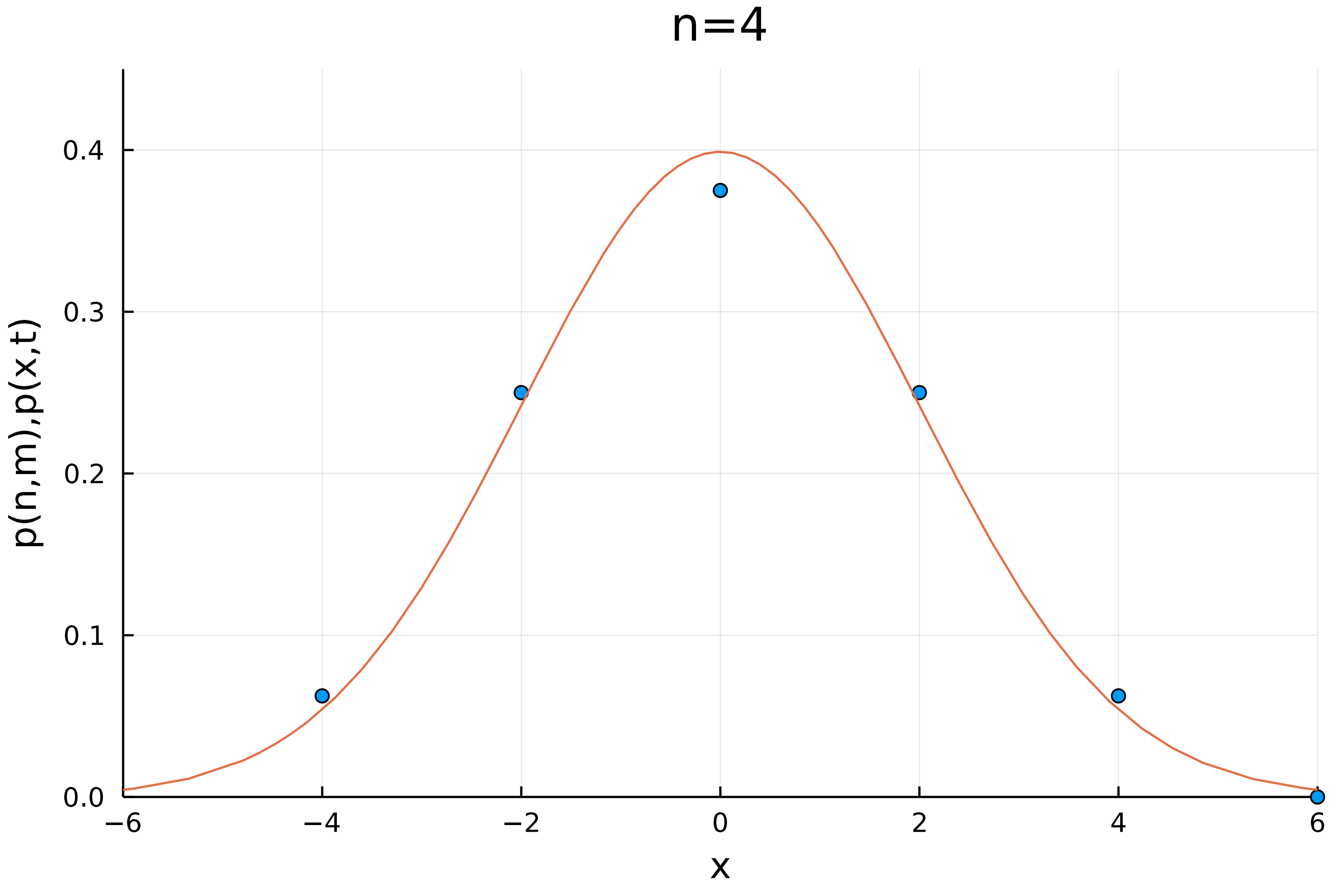
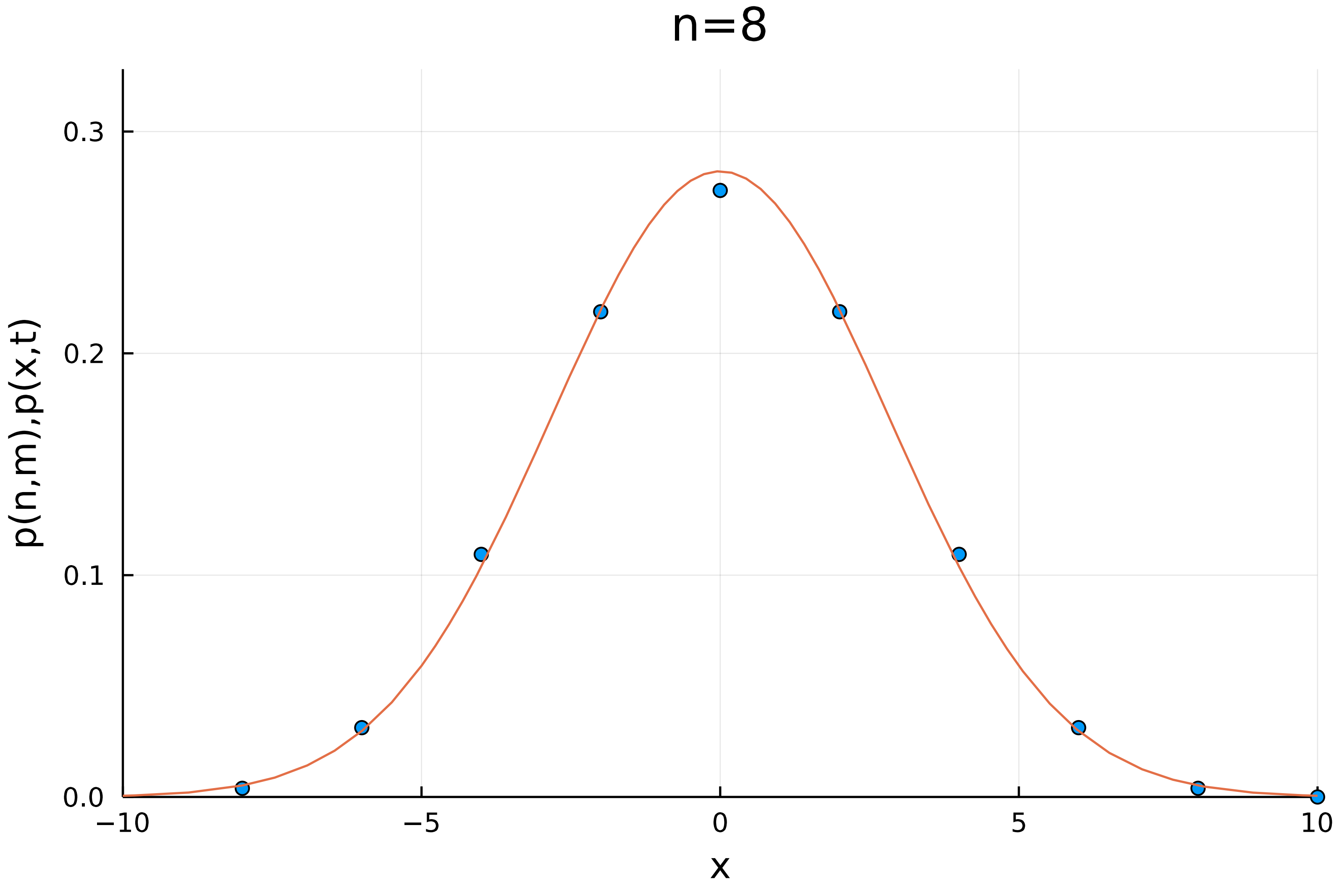

n=0
0 1 0
n=1
0 1/2 0 1/2 0
n=2
0 1/4 0 1/2 0 1/4 0
n=3
0 1/8 0 3/8 0 3/8 0 1/8 0
n=4
0 1/16 0 4/16 0 6/16 0 4/16 0 1/16 0

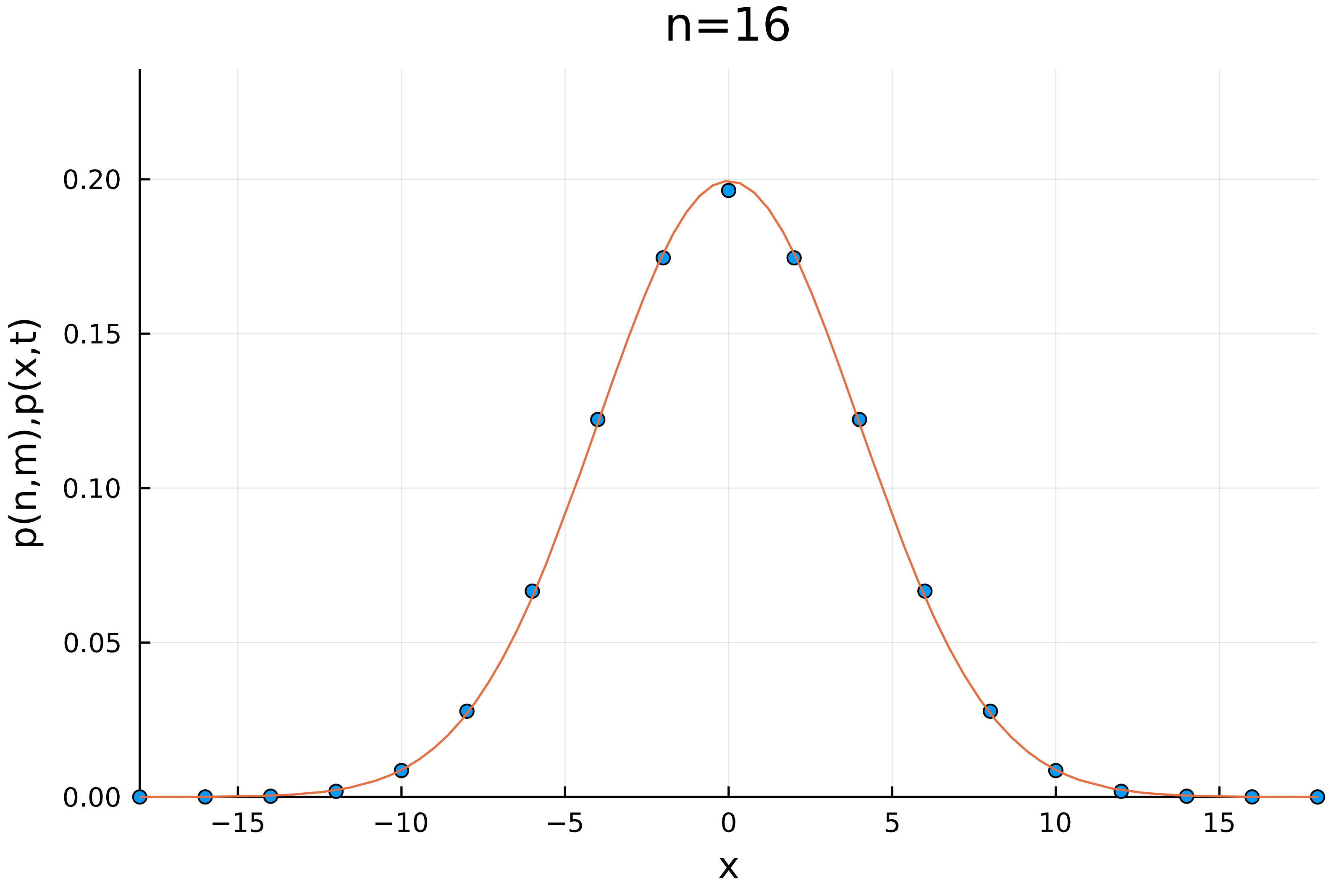

n=2
p(n,m),p(x,t)
0.0
0.1
0.2
0.3
0.4
0.5
−4
−2
0
2
4
x

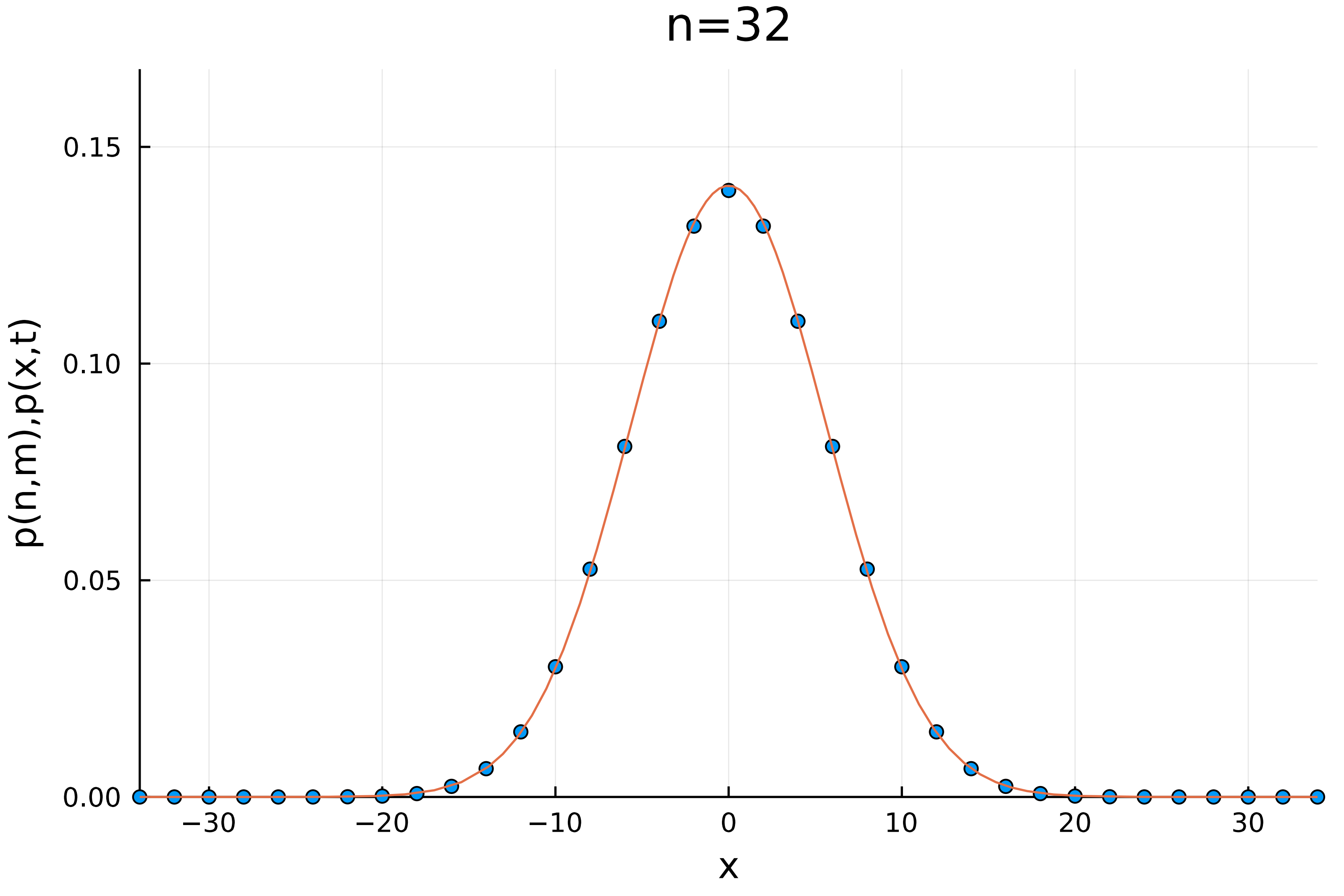


n=4
p(n,m),p(x,t)
0.0
0.1
0.2
0.3
0.4
−6
−4
−2
0
2
4
6
x



n=8
p(n,m),p(x,t)
0.3
0.2
0.1
0.0
−10
−5
0
5
10
x


n=16
p(n,m),p(x,t)
0.00
0.05
0.10
0.15
0.20
−15
−10
−5
0
5
10
15
x


n=32
p(n,m),p(x,t)
0.15
0.10
0.05
0.00
−30
−20
−10
0
10
20
30
x