AMSI SS2023: GEOMETRIC GROUP THEORY — PRE-QUIZ

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Instructions. The following questions assume you have taken a first course in abstract algebra (group theory), and had some exposure to proofs (eg. from taking discrete mathematics, combinatorics, or analysis). We expect the quiz to take around an hour.

Notation:
- If G is a group and H a subgroup of G, let $[G\colon H]$ denote the *index* of H in G, *i.e.* the number of left (or right) cosets of H.
- Let 1_G be the identity element of G, and $|G|$ the size (or order) of G.
- Let C_n denote the cyclic group of order $n \in \mathbb{N}_+$.

1. (a) Let G, H be groups and $f\colon G \to H$ a homomorphism, and let
$$N = \{g \in G \mid f(g) = 1_H\}.$$
Prove that N is a normal subgroup of G.

   (b) What is the subgroup N usually called?

2. Prove that if H is a subgroup of G and $[G : H] = 2$ then H is a normal subgroup.

3. Let G be a finite group with $|G|$ a prime number. Prove that any group homomorphism $\varphi\colon G \to H$ is either the trivial homomorphism[1] or a one-to-one map.

4. Give a definition of a *metric space*.

5. How many non-isomorphic trees[2] with 5 nodes (vertices) are there?

6. (a) Prove that if G is a finite group, then G has a generating set of size at most $\log_2 |G|$.[3]

   (b) Which of the following groups G (all of size 8) show that the bound $\log_2 |G|$ in part (a) is sharp?
   - the quaternion group[4]
   - the dihedral group of order 8 (the group of symmetries of a square)
   - C_8
   - $C_2 \times C_4$
   - $C_2 \times C_2 \times C_2$

END OF PRE-QUIZ For Solutions click here

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[1] $\varphi(g) = 1_H$ for all $g \in G$

[2] https://en.wikipedia.org/wiki/Tree_(graph_theory)

[3] Hint: Suppose $A = \{g_1, \dots g_m\}$ is a *minimal* generating set for G. This means $A \setminus \{g_i\}$ does not generate G for each i. Is $g_2 \in \langle g_1 \rangle$? If not, what is the index of $\langle g_1 \rangle$ in G (at least)? Show that $|G_{i+1}| \geqslant 2|G_i|$ where $G_i = \langle g_1, \dots, g_i \rangle$.

[4] https://en.wikipedia.org/wiki/Quaternion_group