Student number

AMSI Summer School Diagnostic Quiz, 2023

School of Mathematics and Statistics

Random Matrix Theory in Quantum Information

Submission deadline:

This assignment consists of 11 pages (including this page)

Question 1

Consider five matrices $A, B \in \mathbb{C}^{N\times N}$, $C \in \mathbb{C}^{N\times M}$, $D \in \mathbb{C}^{m\times N}$, and $E \in \mathbb{C}^{M\times M}$ of dimensions $N \times N$, $N \times M$, $M \times N$ and $M \times M$, respectively. Additionally, the matrix E should be invertible. Show the following three identities for the determinant.

(a) $\det[AB] = \det[A]\det[B]$ (Prove this by the Leibniz formula of the determinant),

(b) $\det\begin{bmatrix} A & C \\ D & E \end{bmatrix} = \det[A - CE^{-1}D]\det[E]$ (Prove this by using the identity (a)),

(c) $\det[\mathbf{1}_N - CD] = \det[\mathbf{1}_M - DC]$ (Prove this by using identity (b)).

Question 2

Let $n \leq p$ be two positive integers and $X \in \mathbb{C}^{n\times p}$ be a complex $n \times p$ matrix with $XX^\dagger \in \mathbb{C}^{n\times n}$ being invertible. Prove that there are two unitary matrices $U \in \mathrm{U}(n)$, $V \in \mathrm{U}(p)$, and a matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n & 0 & \dots & 0 \end{bmatrix} \quad \text{with } \lambda_1, \dots, \lambda_n \geq 0$$

and where the number of last columns (that equal to zero) is equal to $p - n$, such that

$$X = U\Lambda V^\dagger.$$

You can employ the eigenvalue decomposition of Hermitian matrices. (This matrix decomposition is called singular value decomposition.)

Question 3

(a) Let $a, b \in \mathbb{C}$ be two fixed complex numbers and a has a positive real part $\text{Re}(a) > 0$. Prove the following integral:

$$\int_{-\infty}^{\infty} \exp[-ax^2 + 2bx]dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\left[\frac{b^2}{a}\right],$$

where $\sqrt{a}$ is the principal value of the square root of a mean it has a branch cut along the negative real line.

(b) Let $A \in \mathbb{R}^{3\times 3}$ be an invertible 3×3 real matrix. Compute the Gaussian integral

$$I(A) = \int_{\mathbb{R}^3} \exp[-x^T A^T Ax]d^3x$$

where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is a three dimensional column vector and the volume element is $d^3x = dx_1dx_2dx_3$.

Question 4

We define the tensor product of two matrices $A \in \mathbb{C}^{N\times N}$ and $B \in \mathbb{C}^{M\times M}$ as follows

$$C = A \otimes B \in \mathbb{C}^{NM\times NM} \text{ with matrix entries } C_{\alpha\beta} = C_{(a,\tilde{a})(b,\tilde{b})} = A_{ab}B_{\tilde{a}\tilde{b}}$$

with the multi-indices $\alpha = (a, \tilde{a})$ and $\beta = (b, \tilde{b})$ that run over a 2-dim lattices of integers given by $a, b = 1, \ldots, N$ and $\tilde{a}, \tilde{b} = 1, \ldots, M$.
Prove the following properties of the tensor product

(a) $\operatorname{tr} C = \operatorname{tr}(A)\operatorname{tr}(B)$,

(b) $(A_1 + A_2) \otimes (B_1 + B_2) = A_1 \otimes B_1 + A_2 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_2$ for all $A_1, A_2 \in \mathbb{C}^{N\times N}$ and $B_1, B_2 \in \mathbb{C}^{M\times M}$,

(c) $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1A_2) \otimes (B_1B_2)$ for all $A_1, A_2 \in \mathbb{C}^{N\times N}$ and $B_1, B_2 \in \mathbb{C}^{M\times M}$.

End of Assignment