FOURIER ANALYSIS AND DISTRIBUTION THEORY - QUIZ

The basic prerequisite for this paper is simply that you have completed a standard undergraduate paper in real analysis. Essentially this means that, for instance, you understand how to define limits of series and functions, can explain the difference between continuous and uniformly continuous functions, and know the key theorems from real analysis (i.e. completeness of the real line, the Bolzano-Weierstrass theorem, Mean-Value theorem, etc). It is also helpful if you have had some exposure to (normed) vector spaces, and analysis in higher dimensions.

Feel free to get in contact before the summer school if you are unsure about the background required, or would like to be directed to suitable background reading, we are happy to help!

Questions

The following questions should give you a rough idea of the background required. If you get stuck, try think on it for awhile. If you find that you are still unable to make progress, the solutions can be found below.

Question 1. Let $A, B, C \subset \mathbb{R}$. Suppose $f_j : A \to B$, $f : A \to B$, $g : B \to C$ and

- $f_j \to f$ uniformly on A,
- g is uniformly continuous on B.

a) Show that $g \circ f_j \to g \circ f$ uniformly on A. (Here we let $(g \circ f)(x) = g\big(f(x)\big)$ denote the *composition* of g and f.)

b) Can we drop either of the uniformity assumptions and still maintain the conclusion in a)? Give either a proof or a counterexample.

Question 2. Use the Weierstrass M-Test to prove that the following series converge uniformly in the given domains (that is that the sequence of their partial sums converges uniformly).

a)
$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad x \in [-1, 1]$$

b)
$$f(x) = \sum_{n=0}^{\infty} e^{nx}, \qquad x \in (-\infty, -1]$$

Question 3. Let $n \in \mathbb{N}$ and define
$$a_n = \int_0^n \frac{1}{(1+x^2)^3} dx.$$
Show that $(a_n)_{n \in \mathbb{N}}$ is a convergent sequence.

Question 4. Define the function $f : \mathbb{R} \to \mathbb{R}$ as
$$f(t) = \begin{cases} \exp(-\frac{1}{t}), & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

a) Show that f is infinitely differentiable, thus $f \in C^{\infty}(\mathbb{R})$. Is f analytic?

b) Define $g(x) = f(1 - |x|^2)$. Show that $g \in C^{\infty}(\mathbb{R})$ and $\operatorname{supp} g \subset \{|x| \leqslant 1\}$. Thus there exists smooth functions which are compactly supported.

Question 5. Use the mean value theorem to show that $\sqrt{4+5x} < \frac{5x+13}{6}$ for all $x > 1$.

Question 6. Show that the derivative $\frac{d}{dx}$ is a linear operator on $C^\infty(\mathbb{R})$.

Question 7. Suppose $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed vector spaces. Show that the Cartesian product

$$X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1,\ x_2 \in X_2\}$$

is a normed vector space with norm

$$\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2, \qquad x_1 \in X_1,\ x_2 \in X_2.$$

Solution 1.

a) Set $\epsilon > 0$ and consider $|g(f_j(x)) - g(f(x))|$. Since g is uniformly continuous there is some δ so that if $|y - b| \leq \delta$
$$|g(y) - g(b)| \leq \epsilon \quad \text{for every } b \text{ in } B$$
Since both $f_j(x)$ and $f(x)$ are in B we know that if $|f_j(x) - f(x)| \leq \delta$ then
$$|g(f_j(x)) - g(f(x))| \leq \epsilon.$$
Since $f_j \to f$ uniformly there is a J dependent only on δ so that $|f_j(x) - f(x)| \leq \delta$. Therefore $g(f_j(x)) \to g(f(x))$ uniformly.

b) We used both instances uniformly actively in our proof so we should suspect that we cannot remove either. This is indeed the case, we can see it through counter examples. Let $g(x) = x$ then
$$|g(x) - g(a)| = |x - a|$$
and so by setting $\delta = \epsilon$ we can see $g(x)$ is uniformly continuous on any subset of $\mathbb{R}$. The sequence of functions $f_j(x) = |x|^j$ is pointwise but not uniformly convergent (check this by looking at what happens to the rate of converge near 1) to
$$f(x) = \begin{cases} 0 & |x| < 1 \\ 1 & |x| = 1 \end{cases}$$
on $[-1, 1]$. Now with $A = B = C = [-1, 1]$ we have $g(f_j(x)) = |x|^j$ so clearly we need the uniform convergence of the $f_j(x)$. On the other hand $f_j = \frac{1+jx}{j}$ is convergent uniformly to $f(x) = x$ on the whole of $\mathbb{R}$ (since $f_j(x) - x = \frac{1}{j}$). Let $g(x) = x^2$. This is continuous on $\mathbb{R}$ (as it is a monomial) but if it were uniformly continuous that would imply there was some δ so that $|x - a| \leq \delta$ garantees that
$$|x^2 - a^2| = |x - a||x + a| \leq \epsilon \quad \text{for any } a$$
To see this is impossible consider $a_k = k$ and $x = a_k + \delta/2$. Then this implies that
$$\frac{\delta}{2}(2k + \delta/2) \leq \epsilon$$
which has a contradiction if $k \to \infty$. Now $g(f_j(x)) = \frac{1+2jx+j^2x^2}{j^2}$. Consider
$$|g(f_j(x)) - g(f(x))| = \left|\frac{1 + 2jx + j^2x^2}{j^2} - x^2\right| = \left|\frac{1 + 2jx}{j^2}\right|$$
If this convergence is uniform there is some J so that for all x
$$\frac{|1 + 2Jx|}{J^2} \leq \epsilon$$
again consider the sequence $x_k = k$. As $k \to \infty$ the left hand side diverges to ∞ so cannot be less than ϵ. Therefore we need both uniformity assumptions.

Solution 2.

a) Since $|x| \leq 1$
$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$$
therefore set $M_n = \frac{1}{n^2}$. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (in fact it is equal to $\pi^2/6$) so by the M test the series is uniformly convergent on $[-1, 1]$.

b) The function e^x is increasing so if $x_1 \leq x_2$, $e^{x_1} \leq e^{x_2}$. Therefore for $x \leq -1$, $e^x \leq e^{-1}$. That is
$$|e^{nx}| \leq \left(\frac{1}{e}\right)^n$$

So set $M_n = \left(\frac{1}{e}\right)^n$. Since $e > 1$, $1/e < 1$ so

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$$

is a converging geometric series. Therefore by the M test, $\sum_{n=0}^{\infty} e^{nx}$ converges uniformly in $(-\infty, -1)$.

Solution 3. Notice that

$$a_{n+1} = a_n + \int_n^{n+1} \frac{1}{(1+x^2)^3} dx$$

and since the integrand is positive (and therefore the integral is positive) we have

$$a_{n+1} \geq a_n$$

so this is a monotone increasing sequence. To show it is convergent all we need to do is show that it is bounded. Notice that $1 + x^2 \geq 1$, so

$$\frac{1}{(1+x^2)^3} \leq \frac{1}{1+x^2} \cdot 1 \cdot 1$$

so

$$a_n \leq \int_0^n \frac{1}{1+x^2} dx = [\arctan(x)]_0^n \leq \frac{\pi}{2}$$

since $\arctan(x) \leq \frac{\pi}{2}$ for any x. Therefore this is a bounded monotone sequence and is convergent.

Solution 4.

a) It is clear that f is infinitely differentiable when $t < 0$ and $t > 0$. Thus we only have to check that the derivatives $(\frac{d}{dt})^N f(0)$ exist for every $N \in \mathbb{N}$. Since f vanishes for $t < 0$, it is clear that the only possible value for the derivative at zero is 0. Thus it suffices to prove that

$$\lim_{t \to 0, t > 0} \frac{0 - (\frac{d}{dt})^N f(t)}{t} = 0.$$

But this follows from an inductive argument, together with the standard property that the exponential grows quicker than any polynomial, thus for any $M \in NN$ there exists a constant $C_M > 0$ such that for every $x \geqslant 0$

$$e^x \geqslant C_M x^M.$$

This inequality can be rearranged to give $e^{-\frac{1}{t}} \leqslant C_M^{-1} t^M$, from which the limit quickly follows.

Although f is a smooth function (i.e. C^∞) it is not analytic, this can be seen for instance by expanding f as a Taylor series about $t = 0$, or alternatively recalling that any analytic function which vanishes in an open set must in fact be identically zero.

b) The fact that $g \in C^\infty(\mathbb{R})$ follows directly from the chain rule. To check the support condition, we simply observe that if $|x| \geqslant 1$, then $1 - |x|^2 \leqslant 0$ and so $f(1 - |x|^2) = 0$. Thus $g(x) = 0$ whenever $x \notin \{|x| \leqslant 1\}$ and hence $\operatorname{supp} g \subset \{|x| \leqslant 1\}$.

Solution 5. Let $f(x) = \sqrt{4 + 5x}$ for all $x \geq 1$. Then

$$f'(x) = \frac{5}{2\sqrt{4+5x}} = \frac{5}{2\sqrt{4+5x}},$$

and note that this is strictly decreasing on $[1, \infty)$. Now, fix $x \in \mathbb{R}$ such that $x > 1$. We apply the mean value theorem to f on the interval $[1, x]$. Then, since $f(1) = 3$, there exists $c \in (1, x)$ such that

$$\sqrt{4+5x} - 3 = f'(c) \cdot (x - 1).$$

Since $f'(1) = \frac{5}{6}$ and $f'(c) < f'(1)$ for $c > 1$ we conclude that

$$\sqrt{4+5x} - 3 < \frac{5}{6} \cdot (x - 1).$$

Rearranging terms then provides the desired inequality.

Solution 6. This simply requires that we check that for every $f, g \in C^\infty(\mathbb{R})$ and $\lambda \in \mathbb{R}$ we have

$$\frac{d}{dx}(f + \lambda g) = \frac{d}{dx} f + \lambda \frac{d}{dx} g$$

but this is a consequence of the definition of the derivative, together with the algebra property of the limit.

Solution 7. We just have to check that $\|\cdot\|$ satisfies the properties of a norm. If $x = (0, 0)$ then

$$\|x\| = \|0\|_1 + \|0\|_2 = 0.$$

On the other hand, if $x = (x_1, x_2) \in X_1 \times X_2$ and $\|x\| = 0$ then $0 = \|x_1\|_1 + \|x_2\|_2$. Since both $\|\cdot\|_1$ and $\|\cdot\|_2$ are non-negative and norms, we see have $\|x_1\|_1 = 0 \Rightarrow x_1 = 0$ and $\|x_2\|_2 = 0 \Rightarrow x_2 = 0$. Hence $\|x\| = 0$ if and only if $x = (0, 0)$. Thus $\|\cdot\|$ is non-degenerate on $X_1 \times X_2$.

To verify the (absolute) homogeneity and triangle inequality, we again note that as $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms we have for any $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $\alpha \in \mathbb{C}$ (assuming X_j are vector spaces over $\mathbb{C}$)

$$\|\alpha x\| = \|\alpha x_1\|_1 + \|\alpha x_2\|_2 = |\alpha|(\|x_1\|_1 + \|x_2\|_2) = |\alpha| \, \|x\| .$$

and

$$\|x + y\| = \|x_1 + y_1\|_1 + \|x_2 + y_2\|_2 \leq \|x_1\|_1 + \|y_1\|_1 + \|x_2\|_2 + \|y_2\|_2 = \|x\| + \|y\| .$$

Therefore $\|\cdot\|$ is a norm on $X_1 \times X_2$.