MEAN CURVATURE FLOW - INTRODUCTION AND RESULTS IN VARIOUS SETTINGS QUIZ

GLEN AND VALENTINA-MIRA WHEELER

1. WHAT IS THIS THING?

Here are the instructions we received:

> The AMSI Summer School often attracts students with varying skill-sets from different cognate disciplines, such as physics, engineering, biosciences, geosciences as well as the more conventional mathematic departments. After the success of recent years, we would also like to include a short quiz that students can take to self-assess whether they are properly equipped with the skills to take the course. This quiz need only be brief, but should include questions that give students a sense of what kind of foundational skills are required in order to competently take on your course. This may also be useful for you to establish a baseline skills level for students undertaking your course. Please keep in mind that the Summer School is primarily targeted towards Honours students.

What a great idea! We plan our course to be broadly accessible to students from a variety of backgrounds. The essential things are *multivariate differentiation and integration* and *differential equations*. Some familiarity with parabolic PDE may be helpful. But we will spend some time on background and prerequisite material in the beginning of the subject. If you are worried about not having enough or the right background, just let us know. We can help and are in general very happy to give advice. Especially if you write to us before the subject, we can give you background material to read if desired.

2. THE THING

Here are some questions that should hopefully look familiar, or possibly doable. Note that there are only a few questions, but each question has many parts. These are presented for you to chew on and relax with, maybe even to learn from. If it looks like too much, just take on the first couple of parts of each question. The questions are patient, they won't be going anywhere.

1. (Divergence Theorem in Euclidean Space) Let S be a bounded open subset of $\mathbb{R}^n$ and suppose ∂S is C^1_{loc}. Let $X : S \to \mathbb{R}^n$ be a vector field on S. Suppose each component function $X^i \in C^1(S)$. The divergence theorem says that

$$\int_S \text{div}X\, dx = \int_{\partial S} X \cdot \nu \, dS\,.$$

Here ν is the unit normal to S pointing away from the interior of S, we use superscripts to denote the i-th component of vectors, and the central dot denotes the dot product of vectors.

Use the divergence theorem to answer prove the following:

(a) (Gauss-Green) Let $u : S \to \mathbb{R}$ be a function in $C^1_{loc}(S) \cap C^0(\overline{S})$. Then

$$\int_S u_{x_i}\, dx = \int_{\partial S} u\nu^i \, dS \qquad \text{for any } i \in \{1, \dots, n\}.$$

(b) (Integration by Parts - One Component) Let $u, v : S \to \mathbb{R}$ be functions in $C^1_{loc}(S) \cap C^0(\overline{S})$. Then

$$\int_S u_{x_i} v\, dx = -\int_S u v_{x_i}\, dx + \int_{\partial S} uv\nu^i \, dS \qquad \text{for any } i \in \{1, \dots, n\}.$$

(c) (Green's First Identity) Let $u : S \to \mathbb{R}$ be a function in $C^2_{loc}(S) \cap C^1(\overline{S})$. Then

$$\int_S \Delta u \, dx = \int_{\partial S} Du \cdot \nu \, dS \, .$$

(d) (Green's Second Identity) Let $u, v : S \to \mathbb{R}$ be functions in $C^2_{loc}(S) \cap C^1(\overline{S})$. Then

$$\int_S v\Delta u \, dx = \int_{\partial S} vDu \cdot \nu \, dx - \int_S Du \cdot Dv \, dx \, .$$

(e) (Green's Third Identity) Let $u, v : S \to \mathbb{R}$ be functions in $C^2_{loc}(S) \cap C^1(\overline{S})$. Then

$$\int_S u\Delta v - v\Delta u \, dx + \int_{\partial S} uDv \cdot \nu - vDu \cdot \nu \, dS \, .$$

2. (Chain rule) Consider the function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by

$$f(x) = \sin(\text{abs}(x)) \, .$$

(I use the notation $\text{abs}(x) = |x|$. It helps with applications of the chain rule, and is fun to say to boot.)

(a) Calculate $\Delta\text{abs}(x)$ using the chain rule.

(b) Calculate

$$\int_{B_1(0)} \Delta f(x) \, dx$$

by using the expression for $\Delta f(x)$ found earlier and evaluating the integral term by term.

(c) Calculate

$$\int_{B_1(0)} \Delta f(x) \, dx$$

by first applying the divergence theorem, and evaluating the boundary integral. Which method did you find easier?

3. (Blowup) We say that an evolution equation $\dot{u}(t) = f(u(t), t)$ (here $\dot{u}$ denotes the time derivative) *blows up* if the solution u exists only locally in time (that is, for short times) but not globally in time. In this question we ask you to investiage this phenomenon for three simple ODEs.

(a) For $u(t) \in \mathbf{R}$, suppose

$$\dot{u} = u \, .$$

Solve for u. Prove that the solution is defined globally in time and grows exponentially as $t \to +\infty$. Worst-case exponential growth is a typical feature of well-posed (linear) evolution equations.

(b) For $n > 2$, $u(t) \in \mathbf{R}$, suppose

$$\dot{u} = \frac{1}{n} u^3 \, .$$

Solve for u with $u(0) = u_0 > 0$. This equation is called a *Ricatti equation*, and we will see it in our study of mean curvature flow. Prove that the solution exists at most for a finite amount of time T, and that $u(t) \to \infty$ as $t \to T$. In general, nonlinearities that grow super-linearly in u can lead to blowup.

(c) For $n > 2$, $u(t) \in \mathbf{R}$, suppose

$$\dot{u} = -u + u^2 \, .$$

Solve for u. Show that if $0 < u(0) < 1$, then the solution exists globally in time. If $u(0) > 1$, show that the solution blows up at some time $T < \infty$ (find it). This is a simple example where we have global existence of solutions with small initial data and only local existence of solutions with large initial data. This type of behavior also occurs in many PDEs; for example, we will see for mean curvature flow that data near balls always blows up, whereas data near planes does not (in senses to be made precise later).

4. (Weak maximum principle) Consider $u \in C^2(S)$, where S is open and bounded.

(a) First, let's start with the Laplace equation. Suppose we have
$$\Delta u(x) = 0$$
for all $x \in S$. We wish to show that
$$\max_{x \in \overline{S}} u(x) = \max_{x \in \partial S} u(x)\,.$$
For this first one, we will step you through one possible approach. We will use a helper function. Let $\varepsilon > 0$ and define
$$u_\varepsilon(x) = u(x) + \varepsilon |x|^2\,.$$
Show that $\Delta u_\varepsilon(x) > 0$.

(b) Suppose that we have a critical value $u_\varepsilon(x_0)$ for u_ε. This means that $Du_\varepsilon(x_0) = 0$. If this critical value is a maximum, then
$$(D^2 u_\varepsilon(x_0)) \leq 0\,.$$
Recall that $D^2 u_\varepsilon$ (the Hessian of u) is a matrix, so this means that the Hessian of u is negative semi-definite. That is,
$$(D^2 u_\varepsilon(x_0))(z, z) \leq 0$$
for every $z \in \mathbb{R}^n$. Show that this means $\Delta u_\varepsilon(x_0) \leq 0$.

(c) Combine the previous two steps to conclude that the maximum of the helper function can only occur on the boundary.

(d) Now we estimate from two directions. First, observe that the maximum of the helper function is strictly bigger than the maximum of the original function, because we added on something non-negative to it. Second, the maximum of the helper function on the boundary is bounded by
$$\max_{x \in \partial S} u(x) + \varepsilon \max_{x \in \partial S} |x|^2\,.$$
Using these two estimates, write an estimate for the maximum of the original function u that involves ε.

(e) Finally, use the hypothesis that S is bounded to conclude the original statement. Note that you may have to use the following fact: Let a, b be two real numbers. Suppose that for all $\varepsilon > 0$,
$$a < b + \varepsilon\,.$$
Then $a \leq b$. (You can refer to this as 'a property of the real numbers'.)

(f) Now it is time to improve our maximum principle. Use the method above as inspiration to show the following. Let $V \in C^0(S; \mathbb{R}^n)$ be a bounded vector field on S. Consider $u \in C^2(S)$, where S is bounded, satisfying
$$\Delta u(x) + \langle V(x), Du(x) \rangle = 0$$
for all $x \in S$. Then
$$\max_{x \in \overline{S}} u(x) = \max_{x \in \partial S} u(x)\,.$$

(g) We added a gradient term to our PDE, and now we will add an inhomogeneity. Use again the method above as inspiration to show the following. Let $V \in C^0(S; \mathbb{R}^n)$ be a bounded vector field on S, and $f \in C^0(S)$ be a continuous function on S. Consider $u \in C^2(S)$, where S is bounded, satisfying
$$\Delta u(x) + \langle V(x), Du(x) \rangle - f(x) = 0$$
for all $x \in S$.
Suppose $f(x) \geq 0$ for all $x \in S$. Then
$$\max_{x \in \overline{S}} u(x) = \max_{x \in \partial S} u(x)\,.$$

(h) The method fails if f is not non-negative. Give an example solution to a PDE where f attains negative values and the claim is false. *Hint: We are just talking about the maximum principle here, not the minimum principle. Consider* $n = 1$ *and the PDE* $\Delta u + 1 = 0$.

(i) Finally, we add a zero-order term (for example, a source). Use again the method above as inspiration to show the following. Let $V \in C^0(S;\mathbb{R}^n)$ be a bounded vector field on S, and $f \in C^0(S)$ be a continuous function on S. Let also $c \in C^0(S)$ be a continuous function on S. Consider $u \in C^2(S)$, where S is bounded, satisfying

$$\Delta u(x) + \langle V(x), Du(x)\rangle + c(x)u(x) - f(x) = 0$$

for all $x \in S$.
Suppose $f(x) \geq 0$, and $c(x) \leq 0$ for all $x \in S$. Then

$$\max_{x\in\overline{S}} u(x) = \max\{0, \max_{x\in\partial S} u(x)\}\,.$$

(j) For each maximum principle proved above, comment on what needs to change in order to obtain analogous ‘minimum principles’ (or lack thereof in the case of (h)).

For quiz solutions, click here