Machine Learning in Financial Mathematics
Pre-course Quiz

Problem 1. Let $f(x, y) = ye^x$. Find its Taylor expansion at $(0, 0)$ up to the second order.

Problem 2. Find $x(t)$ that satisfies the following ordinary differential equation:

$$dx = \sin(t)x(t)\, dt \quad \text{and} \quad x(0) = 1.$$

Problem 3. Consider three data points $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) on $\mathbb{R}^2$. Find $\alpha, \beta \in \mathbb{R}$ that minimise the following quantity:

$$\sum_{i=1}^{3} |\alpha + \beta x_i - y_i|^2.$$

Problem 4. Assume that X and Y are independent standard normal random variables. Show that $X + 2Y$ is also a normally distributed random variable. Find its mean and variance.

Problem 5. Let X and Y be two random variables. Explain the difference between the conditional expectations $\mathbb{E}(X \mid Y = y)$ and $\mathbb{E}(X \mid Y)$.

Solutions

Solution to Problem 1.

$$\begin{aligned} f(x,y) &= \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)xy + \frac{\partial^2 f}{\partial y^2}(0,0)y^2\right) \\ &= y + xy + \mathcal{O}(|x,y|^3) \end{aligned}$$

Solution to Problem 2.

$$\frac{dx}{x} = \sin(t)dt \qquad \Longrightarrow \qquad x(t) = Ce^{\int_0^t \sin(u)du} = Ce^{1-\cos(t)}.$$

By our initial condition, $C = 1$ and $x(t) = e^{1-\cos(t)}$.

Solution to Problem 3. Let $f(\alpha, \beta) = \sum_{i=1}^3 |\alpha + \beta x_i - y_i|^2$. This is a quadratic polynomial in α and β. To find the minimising (α, β), we differentiate to obtain the first order conditions:

$$\begin{aligned} \frac{\partial f}{\partial \alpha} &= 2\sum_{i=1}^3 (\alpha + \beta x_i - y_i) = 0, \\ \frac{\partial f}{\partial \beta} &= 2\sum_{i=1}^3 (\alpha + \beta x_i - y_i)x_i = 0. \end{aligned}$$

Therefore,

$$\begin{pmatrix} 3 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1}\begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}.$$

Solution to Problem 4. The moment generating function of $X + 2Y$ is

$$\begin{aligned} \mathbb{E}e^{r(X+2Y)} &= \mathbb{E}e^{rX}\mathbb{E}e^{(2r)Y} \\ &= e^{\frac{1}{2}r^2}e^{\frac{1}{2}(2r)^2} = e^{\frac{1}{2}5r^2} \end{aligned}$$

Therefore, $X + 2Y$ is a normal random variable with mean zero and variance 5.

Solution to Problem 5. Note that $\mathbb{E}(X \mid Y = y)$ is the conditional expectation of X given the event $\{Y = y\}$, and its answer is a non-random value. On the other hand $\mathbb{E}(X \mid Y)$ is the conditional expectation of X given the random variable Y (or, more formally, with respect to the σ-algebra generated by Y), and its answer is a random variable which takes the form of $f(Y)$ for some measurable function f.
For example, if X and Y are independent random variables with $P(Y = 1) > 0$, then $E(XY \mid Y = 1) = E(X \mid Y = 1) = E(X)$, and $E(XY \mid Y) = YE(X \mid Y) = YE(X)$.