Geological fluid dynamics quiz

1. Consider the vector function

$$\mathbf{F} = (x^3, y^3, z).$$

(a) Calculate $\nabla \times \mathbf{F}$ (curl $\mathbf{F}$).
(b) Find a scalar function, $\Phi(x, y, z)$ such that

$$\mathbf{F} = \nabla\Phi.$$

Does such a function Φ exist for any vector function?
(c) Calculate $\nabla \cdot \mathbf{F}$.
(d) Use the divergence theorem to calculate the following surface integral

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S,$$

where ∂V is the surface (both curved surface and the end disks) of the cylinder $0 < z < 1$, $x^2 + y^2 < 1$ and $\mathbf{n}$ is the outward pointing normal.

2. Solve the following ordinary differential equations.

$$\text{(a)} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x}{y^2}, \qquad y(0) = 1,$$

$$\text{(b)} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4, \qquad y(0) = 1, \quad y'(0) = 0.$$

3. (Optional) Write a computer program to solve question 2a using Euler's method in $0 \leq x \leq 1$.
How could you solve question 2b using Euler's method?

4. Consider the heat equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}.$$

(a) If the solution can be written in the form $c(\eta) = c(x, t)$, where $\eta = x/t^{1/2}$, show that

$c(\eta)$ satisfies

$$-\frac{\eta}{2}\frac{\mathrm{d}c}{\mathrm{d}\eta} = \frac{\mathrm{d}^2c}{\mathrm{d}\eta^2}.$$

(b) Hence, or otherwise, show that $c = \mathrm{erfc}[x/(2t^{1/2})]$ is a solution to the heat equation, where erfc is the complementary error function.

Solutions

1.(a)

$$\nabla \times \mathbf{F} = (0,0,0).$$

1.(b)

$$\Phi = x^4/4 + y^4/4 + z^2/2$$

No, it is not always possible to write vector functions as the gradient of a scalar function.

1.(c)

$$\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 1$$

1.(d) By the divergence theorem,

$$\iint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S, = \iiint_V \nabla \cdot \mathbf{F} \, \mathrm{d}V = \int_0^1 \int_0^{2\pi} \int_0^1 (3r^2 + 1) r \, \mathrm{d}r \mathrm{d}\theta \mathrm{d}z = 5\pi/2.$$

2.(a)

$$y = \left(3x + 3x^2/2 + 1\right)^{1/3}.$$

2.(b)

$$y = -2e^x + e^{2x} + 2$$

3. Pick some large integer N say $N = 500$.
Discretize with $x_n = n\delta x$ for $n = 0 ... N$ and $\delta x = 1/N$.
Then $y_0 = 1$ and

$$y_{n+1} = y_n + \left(\frac{1 + x_n}{y_n^2}\right) \delta x$$

For the second order equation, write $z = \mathrm{d}y/\mathrm{d}x$ and

$$\mathrm{d}z/\mathrm{d}x = 4 + 3z - 2y,$$
$$\mathrm{d}y/\mathrm{d}x = z.$$

Then apply Euler's method to this two-dimensional system.

4. (a) Note that

$$\frac{\partial}{\partial t} = -\frac{\eta}{2t} \frac{\mathrm{d}}{\mathrm{d}\eta}, \qquad \frac{\partial}{\partial x} = \frac{1}{t^{1/2}} \frac{\mathrm{d}}{\mathrm{d}\eta}.$$

4.(b) Write $c = \mathrm{erfc}(\eta/2)$ and substitute into the ODE for $c(\eta)$ to verify.