AMSI Summer School 2020

Stochastic Modelling

Pre-course Quiz

1. Suppose A and B are events on the same probability space, use the axioms of probability to show that

   (a) If $A \subset B$, then $\mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A)$;

   (b) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

2. A new surgical procedure is successful with probability p. Assume that this procedure is performed five times and the results are independent of each other.

   (a) What is the probability of five surgical procedures being successful if $p = 0.8$?

   (b) What is the probability that exactly four surgical procedures are successful if $p = 0.6$?

   (c) What is the probability that less than two surgical procedures are successful if $p = 0.3$?

3. Suppose $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with

$$\mathbb{E}[X_i] = \mu \quad \text{and} \quad \mathrm{Var}[X_i] = \sigma^2,$$

   define the random variable

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

   (a) Derive an expression for $\mathbb{E}[\overline{X}_n]$ and $\mathrm{Var}[\overline{X}_n]$.

   (b) For any random variable Y that takes non-negative values only, Markov's inequality says

$$\mathbb{P}(Y \geq a) \leq \frac{\mathbb{E}[Y]}{a} \quad \text{for any constant } a > 0.$$

   Using this result, or otherwise, prove the Tchebyshev's Inequality: If X is a random variable with finite mean $\mathbb{E}[X] = \mu$ and finite variance $\mathrm{Var}[X] = \sigma^2$, then for any constant $k > 0$,

$$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

4. Consider a four-state discrete-time Markov chain defined on the state space $\mathcal{S} = \{0, 1, 2, 3\}$ with transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 1/4 & 3/4 & 0 \end{bmatrix}.$$

   Find $\left[P^{(3)}\right]_{2,2}$, the three-step transition probability that, given the system has started in state 2 at time zero, it will be in state 2 in time 2.

Solutions

1. (a) Note that $\{B \cap A\}$ and $\{B \cap A^c\}$ is a partition of B. We have

$$B = (B \cap A) \cup (B \cap A^c) = A \cup (B \cap A^c),$$

where the second equality follows because $A \subset B$ implies $A \cap B = A$.
As A and $B \cup A^c$ are disjoint, by Axiom 3,

$$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) \quad \Rightarrow \quad \mathbb{P}(B \cap A^c) = \mathbb{P}(B) - \mathbb{P}(A).$$

(b) Note that $A \cap B, A \cap B^c$ is a partition of A: $A = (A \cap B) \cup (A \cap B^c)$. As $A \cap B$ and $A \cap B^c$ are disjoint,

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) \\ \Rightarrow \mathbb{P}(A \cap B^c) &= \mathbb{P}(A) - \mathbb{P}(A \cap B). \end{aligned} \tag{1}$$

Furthermore, $B, A \cap B^c$ is a partition of $A \cup B$: $A \cup B = B \cup (A \cap B^c)$. As B and $A \cap B^c$ are disjoint,

$$\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \cap B^c). \tag{2}$$

Substituting (1) into (2) gives us $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A) - \mathbb{P}(A \cap B)$.

2. (a) Let Y be a random variable counting the number of successful surgical procedures, Then, $Y \sim \text{Bin}(5, p)$, where $p =$ probability of a successful surgical procedure, and

$$\mathbb{P}(Y = 5) = \binom{5}{5}(0.2)^0(0.8)^5 \approx 0.3277.$$

(b) $\mathbb{P}(Y = 4) = \binom{5}{4}(0.4)(0.6)^4 \approx 0.2592.$

(c) $\mathbb{P}(Y < 2) = \mathbb{P}(Y = 0) + \mathbb{P}(Y = 1) = \binom{5}{0}(0.7)^5(0.3)^0 + \binom{5}{1}(0.7)^4(0.3) \approx 0.5282.$

3. (a)

$$\begin{aligned} \mathbb{E}[\overline{X}_n] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}n\mu = \mu, \\ \text{Var}[\overline{X}_n] &= \text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\text{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}. \end{aligned}$$

(b) If X is a random variable with finite mean $\mathbb{E}[X] = \mu$ and finite variance $\text{Var}[X] = \sigma^2$, then for any constant $k > 0$,

$$\begin{aligned} \mathbb{P}(|X - \mu| \geq k\sigma) &= \mathbb{P}\left(\frac{|X - \mu|}{\sigma} \geq k\right) = \mathbb{P}\left(\frac{(X - \mu)^2}{\sigma^2} \geq k^2\right) \\ &= \mathbb{P}(Y \geq a), \quad \text{where } Y = (X - \mu)^2/\sigma^2 \geq 0 \text{ and } a = k^2 > 0, \\ &\leq \frac{\mathbb{E}[Y]}{k^2} = \frac{1}{k^2}, \text{ from above, as } \mathbb{E}[(X - \mu)^2] = \sigma^2. \end{aligned}$$

4.
$$P^3 = \begin{bmatrix} 1/4 & 9/32 & 3/16 & 9/32 \\ 9/64 & 13/64 & 21/32 & 0 \\ 5/64 & 7/32 & 9/64 & 9/16 \\ 5/64 & 7/32 & 9/16 & 9/64 \end{bmatrix}.$$

Therefore $[P^3]_{2,2} = 9/64$.