

# Quiz – Introduction to String Theory

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## 1. Commutator.

The commutator of two objects (you can assume that they are square  $d \times d$ -matrices) is defined as  $[A, B] := AB - BA$ . Show that

- (a)  $[A, B] = -[B, A]$
- (b)  $[A, B + C] = [A, B] + [A, C]$
- (c)  $[AB, C] = A[B, C] + [A, C]B$

We can also define an anticommutator:  $\{A, B\} := AB + BA$ . Decompose  $\{AB, C\}$  like  $[AB, C]$  is decomposed above.

## 2. Euler-Lagrange equation.

Let  $x(t)$  be the position of a point-particle moving in one spatial dimension. Its motion is parameterised by a time coordinate  $t$ . Let  $\dot{x} := \frac{dx}{dt}$ . Given a “Lagrangian”  $L = L(x, \dot{x})$  ( $x$  and  $\dot{x}$  are understood as independent variables), the equation of motion of the particle is given by the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0. \quad (1)$$

Given  $L = \frac{1}{2}\dot{x}^2$ , compute the equation of motion and find the general solution. What is the physical interpretation of the equation and of its solution?

## 3. Conformal field.

Consider a “conformal primary field”  $\phi(z)$  that has the following series expansion:

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^{-n-2}, \quad z \in \mathbb{C}, \phi_n \in \mathbb{C}. \quad (2)$$

Compute

$$\oint_C \phi(z) dz, \quad (3)$$

where  $C$  is a closed contour encircling  $z = 0$ .

## Solutions

1. We verify this by explicit computation of both sides.

(a)

$$\begin{aligned}[A, B] &= AB - BA \\ [B, A] &= BA - AB = -(AB - BA)\end{aligned}\tag{4}$$

(b)

$$\begin{aligned}[A, B + C] &= A(B + C) - (B + C)A = AB + AC - BA - CA \\ [A, B] + [A, C] &= AB - BA + AC - CA\end{aligned}\tag{5}$$

(c)

$$\begin{aligned}[AB, C] &= ABC - CAB \\ A[B, C] + [A, C]B &= A(BC - CB) + (AC - CA)B \\ &= ABC - ACB + ACB - CAB = ABC - CAB.\end{aligned}\tag{6}$$

It is easy to show that the anticommutator is

$$\{AB, C\} = A\{B, C\} - \{A, C\}B.\tag{7}$$

Let us verify this:

$$\begin{aligned}\{AB, C\} &= ABC + CAB \\ A\{B, C\} - \{A, C\}B &= A(BC + CB) - (AC + CA)B \\ &= ABC + ACB - ACB + CAB = ABC + CAB\end{aligned}\tag{8}$$

2. Since  $x$  and  $\dot{x}$  are independent we have

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial \dot{x}} = \dot{x}.\tag{9}$$

The Euler-Lagrange equations are

$$0 - \frac{d}{dt}\dot{x} = -\ddot{x} = -\frac{d^2x}{dt^2} = 0 \quad \rightarrow \quad \ddot{x}(t) = 0.\tag{10}$$

This differential equation is easily solved by integration:

$$\dot{x}(t) = A, \quad x(t) = At + B,\tag{11}$$

where  $A, B \in \mathbb{R}$  are integration constants. The physical interpretation is as follows. If  $x(t)$  is the position of the particle at time  $t$ ,  $\dot{x}$  is its velocity and  $\ddot{x}$  is the acceleration (i.e. change of velocity over time). The equation of motion describes a particle that is not accelerated and moves at a constant velocity  $A$ . Its position at  $t = 0$  is  $x(0) = B$ .

3. The series expansion of  $\phi(z)$  is a Laurent series with poles at  $z = 0$ . Let us write down a few terms:

$$\phi(z) = \dots + \phi_1 z^{-3} + \phi_0 z^{-2} + \phi_{-1} z^{-1} + \phi_{-2} + \phi_{-3} z + \dots \quad (12)$$

The integration contour encircles the poles at  $z = 0$ , so the residue theorem tells us that

$$\oint_C \phi(z) dz = 2\pi i \operatorname{Res}_{z=0} \phi(z). \quad (13)$$

By definition, the residue is the coefficient of  $z^{-1}$  in the Laurent series expansion. Therefore we conclude

$$\oint_C \phi(z) dz = 2\pi i \phi_{-1}. \quad (14)$$