Questions

1. Consider

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha I$$

   (a) Solve the equation.

   (b) What happens if $\alpha > 0$, $\alpha = 0$, or $\alpha < 0$?

2. Consider a population made up of susceptible (S), infected (I) and recovered (R) individuals, with total population $N = S + I + R$. Recovered individuals are immune to reinfection.

   (a) If we ignore births and deaths in the population, what differential equation does this mean must N solve?

   (b) Assume that the rate a susceptible person gets infected is $\beta I/N$ and that the rate an infected person recovers is γ.

   Explain why the governing equations are

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta IS/N$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS/N - \gamma I$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I$$

   (c) Assume that a disease is introduced with $S(0) \approx N$, $I(0) \ll N$ [$I(0)$ much less than N], and $R(0) = 0$. Why can we assume that for early time, $S/N \approx 1$?

   (d) While $S/N \approx 1$, find an approximate solution for I. What condition on β and γ determines whether the outbreak grows?

   (e) Revisiting the assumptions: What assumptions do we make about interactions in the population and the spread of the when we assume that the per-susceptible infection rate is $\beta I/N$ and that the per-infected recovery rate is γ?

   (f) In response to the disease, people begin to reduce their interactions with others. What parameter of the model should change?

3. (a) Modify the system in the previous question to look at the case where recovered individuals are susceptible again.

   (b) As t becomes large, find the equilibrium value of I.

   (c) What condition must hold for the infection to persist? Is this the same as the condition needed for the infection to invade?

4. In a programming language of your choice (answers provided in Python):

(a) Print the first 100 natural numbers in ascending order, but if the number is divisible by 3 print fizz and if the number is divisible by 5 print buzz.

(b) Numerically solve

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -bP(t)$$

for $b = 0.1$ with $P(0) = 10$; and $b = -0.1$ with $P(0) = 1000$, for at least 100 time units.

Solutions

1. (a) $I = Ce^{\alpha t}$

   (b) If $\begin{cases} \alpha > 0 & I \text{ grows exponentially} \\ \alpha = 0 & I \text{ remains constant} \\ \alpha < 0 & I \text{ decays exponentially} \end{cases}$

2. (a) $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$

   (b) The total rate of new infection is the per-susceptible infection rate $\beta I/N$ times the number of susceptibles S. So this explains the $\beta IS/N$ terms in the S and I equations. The total rate of recovery is the per-infected recovery rate γ times the number of infecteds I. This explains the γI terms in the I and R equations.

   (c) Initially S/N is approximately 1. While I is small, S will change slowly. So until either a long time has passed or I is no longer small, we can treat S/N as approximately 1.

   (d) $\frac{\mathrm{d}I}{\mathrm{d}t} = (\beta - \gamma)I$. For growth we need $\beta > \gamma$, or equivalently $\beta/\gamma > 1$.

   (e) Due the the $\beta IS/N$ term, we are assuming that the population is well-mixed, all individuals have a similar rate of interactions. Due to the γI term, we are assuming that infection durations are on average $1/\gamma$, and that the probability of recovering at any time is the same, regardless of how long an individual has been infected.

   (f) β should decrease.

3. (a)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta IS/N + \gamma I$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS/N - \gamma I$$

   (b) Since $\frac{\mathrm{d}I}{\mathrm{d}t} = I(\beta S/N - \gamma)$, we conclude that at large time $S \to \gamma N/\beta$. So $I \to N - \gamma N/\beta$.

   (c) We need $\gamma/\beta < 1$, or $\beta/\gamma > 1$. If we repeat the calculations for a new infection to grow that we did previously, we will again arrive at $\beta/\gamma > 1$.

4. The following are Python snippets.

   (a) A Fizz Buzz solution:

```
end_num = 100

for i in range(1, end_num + 1):  # +1 because 'range' uses
                                   [start, end)
    if i % 15 == 0:
```

```
        print("fizz buzz")
    elif i % 3 == 0:
        print("fizz")
    elif i % 5 == 0:
        print("buzz")
    else:
        print(i)
```

(b) A simulation of the simple DE:

```
import scipy.integrate as scint
import matplotlib.pyplot as pl

# set up parameters
change_rate = -.1  # growth if >0, death if <0
initial_condition = [1000.0]
duration = 100  # days
timespan = (0, duration)

# the differential equation to be solved
def model(t, y):
    return change_rate * y

# solve
sol = scint.solve_ivp(fun=model, t_span=timespan, y0=
                                  initial_condition,
                                  dense_output=True)

# plot
pl.plot(sol.t, sol.y[0], '-k')
pl.xlabel('Time (days)')
pl.ylabel('Population')
pl.show()
```

A better solution would be a for loop over the b and initial condition values, but this is simple.