Quiz for *Algebraic Topology: First Steps in Cohomology*

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1. Prove that for any set S, and any (commutative) ring R, the collection R^S of functions $S \to R$ form an R-module.

2. Prove a function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous in the ε-δ sense if an only if it is continuous in the following sense: for every point x in the preimage $f^{-1}\left[(a,b)\right]$ of an open interval (a,b), there is an open interval around x completely inside the preimage.

3. Prove that given abelian groups A and B, a subgroup $C \subset A$ and a homomorphism $g\colon A \to B$, then there is a homomorphism $h\colon A/C \to B$ with $g = h \circ q$ (for $q\colon A \to A/C$ the quotient map) if and only if g restricted to C is constant at the identity element of B.

Answers on the next page.

1. We need to define an abelian group structure on R^S, so for $f, g \in R^S$, define $f+g$ to be the function given by $(f+g)(s) = f(s)+g(s)$, $\underline{0}$ to be the constant function at $0 \in R$, and $-f$ to be given by $(-f)(s) = -f(s)$. That these give R^S the structure of an abelian group follows from the fact R has an underlying additive abelian group structure. Now we can define the R-module structure in a pointwise fashion as well: $r \cdot f$ is the function defined by $(r \cdot f)(s) = rf(s)$. The axioms for a module are likewise checked pointwise. (This is a little bit cheating: in an assignment question with lots of very routine verification, I'd want to see at least one case written out, preferably not the easiest one; the rest can be covered by a "the other cases follow in the same way".)

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous in the ε-δ sense:

   > For every $x \in \mathbb{R}$, for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

   Now take an open interval, without loss of generality of the form $(z - a, z + a)$ for $a > 0$, and take $x \in f^{-1}[(z - a, z + a)]$. Let $\varepsilon = \min\{|f(x) - (z + a)|, |f(x) - (z - a)|\}$. Notice that $(f(x) - \varepsilon, f(x) + \varepsilon) \subseteq (z - a, z + a)$. By the definition of continuity above there is a $\delta > 0$ such that if $|x - y| < \delta$, or in other words, $y \in (x - \delta, x + \delta)$, then $|f(x) - f(y)| < \varepsilon$. But this means that $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subseteq (z - a, z + a)$. Thus $(x - \delta, x + \delta) \subseteq f^{-1}[(z - a, z + a)]$, as we needed to show.

   Now assume that other definition of continuity. Take arbitrary $x \in \mathbb{R}$, and $\varepsilon > 0$. Consider the preimage $f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$, and notice that there is an open interval $(a, b) \subseteq f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$ with $x \in (a, b)$. Let $\delta = \min\{|x - a|, |x - b|\}$, so that we have $(x - \delta, x + \delta) \subseteq (a, b) \subseteq f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$. So that for if $y \in (x - \delta, x + \delta)$, that is, $|x - y| < \delta$, then $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon)$. But this means that $|f(x) - f(y)| < \varepsilon$, as we needed to show.

3. We will denote the identity element of the groups G by 1_G. Assume first that $g = h \circ q$ for some $h : A/C \to B$. Then as $q(c) = 1_{A/C}$ for all $c \in C$ by construction, we have $g(c) = g(q(c)) = g(1_{A/C}) = 1_B$ for all $c \in C$.

   Conversely, assume that for all $c \in C$, $g(c) = 1_B$. Then, for an arbitrary $x \in A/C$, take $a_x \in A$ so that $q(a_x) = x$, and define $h(x) = g(a_x)$. To show this is well-defined, assume be are also given a'_x such that $q(a'_x) = x$. Then $q(a'_x a_x^{-1}) = xx^{-1} = 1_{A/C}$, so $a'_x a_x^{-1} = c \in C$ (as $C = \ker(q)$ by construction). Thus $a'_x = a_x c$, and $g(a'_x) = g(a_x c) = g(a_x)g(c) = g(a_x) = h(x)$, so $h(x)$ is independent of the choice of a_x.

   We now need to show h is a homomorphism. What we will do is show that $h(xy)^{-1}h(x)h(y) = 1_B$. Take the chosen a_x, a_y and a_{xy} and look at

   $$h(xy)^{-1}h(x)h(y) = g(a_{xy})^{-1}g(a_x)g(a_y) = g(a_{xy}^{-1}a_x a_y)$$

   but we also have

   $$q(a_{xy}^{-1}a_x a_y) = q(a_{xy})^{-1}q(a_x)q(a_y) = (xy)^{-1}xy = 1_{A/C}$$

   so that $a_{xy}^{-1}a_x a_y \in C$, so that $g(a_{xy}^{-1}a_x a_y) = 1_B$ and hence $h(xy)^{-1}h(x)h(y) = 1_B$. If $x = 1_{A/C}$, then $a_{1_{A/C}} \in C$, and so $h(1_{A/C}) = g(a_{1_{A/C}}) = 1_B$. Thus h is a homomorphism.