A short quiz

1. For the following random sample of \( X \): -1, 0, 1, 2, find
   
   (a) sample mean;
   (b) sample standard deviation;
   (c) sample covariance of \( X \) and \( X^2 \).

2. Suppose that the random variable \( Y \) has an \( N(0, 1) \) distribution. Also suppose that the discrete random variable \( S \) satisfies \( P(S = -1) = P(S = 1) = 0.5 \) and that \( S \) and \( Y \) are independent random variables. Let \( X = SY \).
   
   (a) Show that \( \text{Cov}(X, Y) = 0 \);
   (b) Show that \( P(X \in [-1, 1], Y \in [-1, 1]) \neq P(X \in [-1, 1]) \cdot P(Y \in [-1, 1]) \).

3. Compute the Fourier transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{i2\pi xt} f(x)dx \) of the function

\[
f(x) = \begin{cases} 
\cos(x), & \text{if } x \in [-1, 1]; \\
0, & \text{otherwise}.
\end{cases}
\]

Answers

1. (a) The sample mean is \( \bar{X} = 0.5 \);
   (b) sample standard deviation \( \hat{\sigma} = 1.290994 \);
   (c) sample covariance \( \hat{\text{Cov}}(X, X^2) = 1.666667 \).

2. (a) \( E(X) = E(SY) = E(S)E(Y) = 0 \), \( E(Y) = 0 \). Thus \( \text{Cov}(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) = E(SY^2) = E(S)E(Y^2) = 0 \).
   
   (b) Since \( Y^2 = X^2 \):

\[
P(X \in [-1, 1], Y \in [-1, 1]) = P(X^2 \in [0, 1], Y^2 \in [0, 1]) = P(Y^2 \in [0, 1]) = P(Y \in [-1, 1]) \text{ and } P(X \in [-1, 1]) = P(Y \in [-1, 1]).
\]

Thus \( P(X \in [-1, 1]) P(Y \in [-1, 1]) = (P(Y \in [-1, 1]))^2 \).
Since \( 0 < P(Y \in [-1, 1]) < 1 \)

\[
P(X \in [-1, 1], Y \in [-1, 1]) \neq P(X \in [-1, 1]) P(Y \in [-1, 1]).
\]

3.

\[
\hat{f}(t) = \int_{-1}^{1} e^{i2\pi xt} \cos(x)dx = \int_{-1}^{1} (\cos(2\pi xt) + i \sin(2\pi xt)) \cos(x)dx
\]

\[
= \int_{-1}^{1} \cos(2\pi xt) \cos(x)dx = \frac{1}{2} \int_{-1}^{1} (\cos((2\pi t + 1)x) + \cos((2\pi t - 1)x))dx
\]

\[
= \frac{\sin(2\pi t + 1)}{2\pi t + 1} + \frac{\sin(2\pi t - 1)}{2\pi t - 1}.
\]