Questions

1. Consider
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \alpha I$$
   (a) Solve the equation.
   (b) What happens if $\alpha > 0$, $\alpha = 0$, or $\alpha < 0$?

2. Consider a population made up of susceptible (S), infected (I) and recovered (R) individuals, with total population $N = S + I + R$. Recovered individuals are immune to reinfection.
   (a) If we ignore births and deaths in the population, what differential equation does this mean must N solve?
   (b) Assume that the rate a susceptible person gets infected is $\beta I/N$ and that the rate an infected person recovers is γ.
   Explain why the governing equations are
$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta IS/N$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS/N - \gamma I$$
$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I$$
   (c) Assume that a disease is introduced with $S(0) \approx N$, $I(0) \ll N$ [$I(0)$ much less than N], and $R(0) = 0$. Why can we assume that for early time, $S/N \approx 1$?
   (d) While $S/N \approx 1$, find an approximate solution for I. What condition on β and γ determines whether the outbreak grows?
   (e) Revisiting the assumptions: What assumptions do we make about interactions in the population and the spread of the when we assume that the per-susceptible infection rate is $\beta I/N$ and that the per-infected recovery rate is γ?
   (f) In response to the disease, people begin to reduce their interactions with others. What parameter of the model should change?

3. (a) Modify the system in the previous question to look at the case where recovered individuals are susceptible again.
   (b) As t becomes large, find the equilibrium value of I.
   (c) What condition must hold for the infection to persist? Is this the same as the condition needed for the infection to invade?

Solutions

1. (a) $I = Ce^{\alpha t}$

   (b) If $\begin{cases} \alpha > 0 & I \text{ grows exponentially} \\ \alpha = 0 & I \text{ remains constant} \\ \alpha < 0 & I \text{ decays exponentially} \end{cases}$

2. (a) $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$

   (b) The total rate of new infection is the per-susceptible infection rate $\beta I/N$ times the number of susceptibles S. So this explains the $\beta IS/N$ terms in the S and I equations. The total rate of recovery is the per-infected recovery rate γ times the number of infecteds I. This explains the γI terms in the I and R equations.

   (c) Initially S/N is approximately 1. While I is small, S will change slowly. So until either a long time has passed or I is no longer small, we can treat S/N as approximately 1.

   (d) $\frac{\mathrm{d}I}{\mathrm{d}t} = (\beta - \gamma)I$. For growth we need $\beta > \gamma$, or equivalently $\beta/\gamma > 1$.

   (e) Due the the $\beta IS/N$ term, we are assuming that the population is well-mixed, all individuals have a similar rate of interactions. Due to the γI term, we are assuming that infection durations are on average $1/\gamma$, and that the probability of recovering at any time is the same, regardless of how long an individual has been infected.

   (f) β should decrease.

3. (a)

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\beta IS/N + \gamma I$$
$$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS/N - \gamma I$$

   (b) Since $\frac{\mathrm{d}I}{\mathrm{d}t} = I(\beta S/N - \gamma)$, we conclude that at large time $S \to \gamma N/\beta$. So $I \to N - \gamma N/\beta$.

   (c) We need $\gamma/\beta < 1$, or $\beta/\gamma > 1$. If we repeat the calculations for a new infection to grow that we did previously, we will again arrive at $\beta/\gamma > 1$.