Quiz questions.

Q1 A biassed coin comes up Heads 2/3 of the time, and Tails 1/3 of the time. A player tosses this coin until it comes up Tails. Let the random variable X denote the number of Heads before the first Tail.
(a) For each $k \in \{0, 1, 2, \dots\}$, write down the probability that X takes the value k.
(b) Evaluate the expectation of X.

Q2 Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers, and let S be a real number, with $\sum_{n=1}^{\infty} a_n = S$. For each n, let $(a_{n,m})_{m=1}^{\infty}$ be an increasing sequence of positive real numbers converging to a_n. Show that $\sum_{n=1}^{\infty} a_{n,m} \to S$ as $m \to \infty$.

Q3 Find the general solution to the differential equation

$$\frac{dx}{dt} = \frac{x}{(t+1)^2},$$

for $t \geq 0$.

Answers.

Q1 (a) The probability that X takes the value k is $\left(\frac{2}{3}\right)^k \frac{1}{3}$.
(b) The expectation of X is given by the sum

$$\sum_{k=0}^{\infty} k \Pr(X = k) = \frac{1}{3} \sum_{k=0}^{\infty} k \left(\frac{2}{3}\right)^k = \frac{1}{3} \frac{2/3}{(1-2/3)^2} = 2.$$

There are other ways to evaluate the expectation, for instance via the recurrence

$$\mathbb{E}X = \frac{2}{3}(1 + \mathbb{E}X).$$

Q2 We first note that each sum $\sum_{n=1}^{\infty} a_{n,m}$ converges, by the Comparison Test. Fix $\varepsilon > 0$. Choose n_0 so that $\sum_{n=n_0+1}^{\infty} a_n < \varepsilon/2$. For each $n = 1, \dots, n_0$, choose some m_n such that, for $m > m_n$, $a_n - \varepsilon/2n_0 < a_{n,m} \le a_n$. Let $m_0 = \max(m_1, \dots, m_{n_0})$.
For $m > m_0$, we have

$$\sum_{n=1}^{n_0} a_n \ge \sum_{n=1}^{n_0} a_{n,m} > \sum_{n=1}^{n_0} (a_n - \varepsilon/2n_0) = \sum_{n=1}^{n_0} a_n - \varepsilon/2.$$

For any m, we have also

$$0 \le \sum_{n=n_0+1}^{\infty} a_{n,m} \le \sum_{n=n_0+1}^{\infty} a_n < \varepsilon/2.$$

Hence we have, for $m > m_0$,

$$\left| \sum_{n=1}^{\infty} a_{n,m} - \sum_{n=1}^{\infty} a_n \right| \le \left| \sum_{n=1}^{n_0} a_{n,m} - \sum_{n=1}^{n_0} a_n \right| + \left| \sum_{n=n_0+1}^{\infty} a_{n,m} - \sum_{n=n_0+1}^{\infty} a_n \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus $\sum_{n=1}^{\infty} a_{n,m} \to S = \sum_{n=1}^{\infty} a_n$, as required.

Q3 We can solve this equation by separating variables:

$$\int \frac{1}{x}\, dx = \int \frac{1}{(t+1)^2} dt,$$

and so $\ln |x| = \frac{-1}{t+1} + C$. We then obtain the general solution $x = Ae^{-1/(t+1)}$.
We can alternatively solve the differential equation by multiplying through by the integrating factor $e^{-\int (1/(t+1)^2)\, dt} = e^{1/(t+1)}$, obtaining

$$\frac{d}{dt}\left(xe^{1/(t+1)}\right) = e^{1/(t+1)} \left(\frac{dx}{dt} - \frac{x}{(t+1)^2} \right) = 0,$$

and then again the general solution $x = Ae^{-1/(t+1)}$.