A short quiz

1. For the following random sample of X: -1, 0, 1, 2, find
   (a) sample mean;
   (b) sample standard deviation;
   (c) sample covariance of X and X^2.

2. Suppose that the random variable Y has an $N(0,1)$ distribution. Also suppose that the discrete random variable S satisfies $P(S=-1)=P(S=1)=0.5$ and that S and Y are independent random variables. Let $X=SY$.
   (a) Show that $Cov(X,Y)=0$;
   (b) Show that $P\big(X\in[-1,1], Y\in[-1,1]\big)\neq P\big(X\in[-1,1]\big)\cdot P\big(Y\in[-1,1]\big)$.

3. Compute the Fourier transform ($\hat{f}(t)=\int_{-\infty}^{\infty}e^{i2\pi xt}f(x)dx$) of the function

$$f(x)=\begin{cases}\cos(x), & \text{if } x\in[-1,1];\\ 0, & \text{otherwise.}\end{cases}$$

Answers

1. (a) The sample mean is $\bar{X}=0.5$;
   (b) sample standard deviation $\hat{\sigma}=1.290994$;
   (c) sample covariance $\hat{Cov}(X,X^2)=1.666667$.

2. (a) $E(X)=E(SY)=E(S)E(Y)=0$, $E(Y)=0$. Thus $Cov(X,Y)=E((X-E(X))(Y-E(Y)))=E(XY)=E(SY^2)=E(S)E(Y^2)=0$.
   (b) Since $Y^2=X^2$:
   $P(X\in[-1,1],Y\in[-1,1])=P(X^2\in[0,1],Y^2\in[0,1])=P(Y^2\in[0,1])=P(Y\in[-1,1])$ and $P(X\in[-1,1])=P(Y\in[-1,1])$.
   Thus $P(X\in[-1,1])\ P(Y\in[-1,1])=(P(Y\in[-1,1]))^2$.
   Since $0<P(Y\in[-1,1])<1$
   $P(X\in[-1,1],Y\in[-1,1])\neq P(X\in[-1,1])\ P(Y\in[-1,1])$.

3.

$$\hat{f}(t)=\int_{-1}^{1}e^{i2\pi xt}\cos(x)dx=\int_{-1}^{1}(\cos(2\pi xt)+i\sin(2\pi xt))\cos(x)dx$$

$$=\int_{-1}^{1}\cos(2\pi xt)\cos(x)dx=\frac{1}{2}\int_{-1}^{1}(\cos((2\pi t+1)x)+\cos((2\pi t-1)x))dx$$

$$=\frac{\sin(2\pi t+1)}{2\pi t+1}+\frac{\sin(2\pi t-1)}{2\pi t-1}.$$