AMSI SS2020: GEOMETRIC GROUP THEORY – PRE-QUIZ

1. Let n be an integer. Prove that if n^2 is even then n is even.

2. (a) How many binary strings of length n do not contain a factor 11?

   (b) How many binary strings of length n do not contain a factor 11 and have final digit 1?

3. (a) Define the terms *equivalence relation* and *equivalence class*.

   (b) Prove that if G is a group and H is a subgroup, then the left (respectively, right) cosets of H in G are equivalence classes of some equivalence relation.

   (c) Show that H is normal[1] if and only if these two equivalence relations (left, right cosets) are actually the same.

   (d) If H is normal, define the *quotient group* G/H and show that it is a group.

   (e) Give (non-trivial!) examples of G, H and G/H.

   (f) Give an example to show why this definition does not yield a group if H is not a normal.

4. State the *first isomorphism theorem* for groups.

END OF QUIZ

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[1]Definition: H is normal if $g^{-1}hg \in H$ for all $g \in G, h \in H$.

Solutions:

1. Contrapositive. If n is odd then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd. The result is the contrapositive statement.

2. (a) Recursive. If b_n is the number of binary strings of length n without a 11 factor, then $b_0 = 1$ and $b_1 = 2$. A string of length $n \geqslant$ either starts with 0 or 1. If it starts with 0, the next letter can be anything, so there are b_{n-1} possible strings. If it starts with 1, the next letter must be 0, and then anything, so there are b_{n-2} possible strings. So in total $b_n = b_{n-1} + b_{n-2}$. This is the Fibonacci sequence (starting at $1, 2$).

   (b) If c_n is the number of strings without 11 and ending with 1, then $c_0 = 0, c_1 = 1$. For $n \geqslant 2$ the last two digits must be 01 and we have b_{n-2} possible prefixes. So this is also the Fibonacci sequence (starting at $0, 1$).

3. (a) Let S be a set and $\mathscr{R} \subseteq S \times S$. We call $\mathscr{R}$ an *equivalence relation* if it is reflexive ($(s,s) \in \mathscr{R}$ for every $s \in S$), symmetric ($(s,t) \in \mathscr{R}$ implies $(t,s) \in \mathscr{R}$) and transitive ($(r,s),(s,t) \in \mathscr{R}$ implies $(r,t) \in \mathscr{R}$). The *equivalence class* of an element $s \in S$ is then defined as $[s]_{\mathscr{R}} = \{t \in S \mid (s,t) \in \mathscr{R}\}$.

   (b) Define a relation $\mathscr{L}$ on G by $(a,b) \in \mathscr{L}$ if $a^{-1}b \in H$.
   - reflexive: $\forall a \in G, a^{-1}a = 1 \in H$ since H is a subgroup, so $(a,a) \in \mathscr{L}$
   - symmetric: $a^{-1}b \in H$ if and only if $(a^{-1}b)^{-1} = b^{-1}a \in H$ since H is a subgroup.
   - transitive: if $a^{-1}b, b^{-1}c \in H$ then $(a^{-1}a)(b^{-1}c) = a^{-1}c \in H$ since H is a subgroup.

   Then $[a]_{\mathscr{L}} = \{b \in G \mid a^{-1}b \in H\} = \{b \in G \mid b \in aH\} = aH$ is the left coset containing a.

   For the right cosets define $\mathscr{R}$ on G by $(a,b) \in \mathscr{R}$ if $ba^{-1} \in H$, which is an equivalence relation by a similar argument. Then $[a]_{\mathscr{R}} = \{b \in G \mid ba^{-1} \in H\} = \{b \in G \mid b \in Ha\} = Ha$ is the right coset containing a.

   (c) Definition: H is normal if $g^{-1}hg \in H$ for all $g \in G, h \in H$.

   Assume H is normal, then $(a,b) \in \mathscr{L}$ iff $a^{-1}b \in H$ iff $a^{-1}b = h$ for some $h \in H$ iff $ba^{-1} = a(a^{-1}b)a^{-1} = aha^{-1} \in H$ since H is normal, so $(a,b) \in \mathscr{L}$ iff $(a,b) \in \mathscr{R}$, so the two equivalence relations (left, right cosets) are actually the same.

   If H is not normal then $aha^{-1} \notin H$ for some $a \in G, h \in H$ by definition, so $(a, ah) \notin \mathscr{R}$ but $a^{-1}ah \in H$ so $(a, ah) \in \mathscr{L}$ so the two equivalence relations are not the same.

(d) G/H is the set of all (left) cosets with the operation $aH * bH = (ab)H$ (where the group operation on G is denoted by juxtaposition). This is well defined since if you choose different elements to be coset representatives, say $cH = aH$ and $dH = bH$ (so $a^{-1}c, b^{-1}d \in H$) then $b^{-1}a^{-1}cd = b^{-1}hd = b^{-1}dh' \in H$ so $(ab)H = (cd)H$ (here we are using that H is normal).
This is a group since the axioms
  - identity: H
  - inverse: for each aH there is $a^{-1}H$ so that $aH * a^{-1} = (aa^{-1})H = 1.H = H$.
  - associative: (inherited from G)

are satisfied.

(e) A non-trivial example should probably be a non-abelian group: S_3 the set of permutations of $\{1, 2, 3\}$. Its subgroup A_3 of even permutations (in cycle notation $(), (123), (132)$) is normal; this can be checked brute-force using the multiplication table, or by observing that A_3 has index 2 in S_3 and any subgroup of index 2 must be normal: if $g \notin H$ then $gH \neq H$ so $gH = Hg$.

(f) [2] Now take the subgroup $H = \{e, (12)\}$ which is not normal since $(13)(12)(13) = (23)$. Suppose the set of (left) cosets was a group with the multiplication as defined (simply by juxtaposition). There are three left cosets: $H, (13)H = \{(13), (123)\}, (23)H = \{(23), (132)\}$. Now $(13)H * (23)H$ is defined to be $((13)(23))H = (132)H$ but if we instead choose different coset representatives $(123)H * (23)H = (12)H = H$.

4. Let G and H be groups, and let $f : G \to H$ be a homomorphism. Then:
   - the kernel of f is a normal subgroup of G,
   - the image of f is a subgroup of H, and
   - the image of f is isomorphic to the quotient group $G/\ker(f)$.

   In particular, if f is surjective then H is isomorphic to $G/\ker(f)$.

If you got stuck with the group theory questions, try going through any textbook with *Abstract Algebra* in the title. A nice short one is Lauritzen *Concrete Abstract Algebra* Chapter 2. Much more elegant solutions than the above are of course possible.

[2] Convention: I am applying permutations left to right. For example $(13)(12) = (123)$ which in one-line notation (as maps) would be $\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{smallmatrix}$.