QUIZ FOR DIFFERENTIAL GEOMETRY AND SYMMETRY

AMSI SUMMER SCHOOL 2020
LA TROBE UNIVERSITY

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Which of the following statements are true? Prove what you claim.

(1) A symmetric $n \times n$ matrix with real entries has all its eigenvalues real.
(2) Any finite-dimensional vector space has a unique basis.
(3) Let S be a finite orthogonal set of vectors in an inner-product vector space $(V, \langle \cdot, \cdot \rangle)$. Then S is linearly independent.
(4) The function $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x^2 + y, 1 + x + y)$, has a local inverse around $F(0, 0) = (0, 1)$.
(5) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous function for which all partial derivatives exist at $(0, 0, 0) \in \mathbb{R}^3$. Then, F is differentiable at $(0, 0, 0)$.
(6) The equation $z^2 = x^2 + y^2$ defines a differentiable surface in $\mathbb{R}^3$.
(7) There exists a differentiable surface in $\mathbb{R}^3$ which has some points with zero Gauss curvature, and some other points with positive Gauss curvature.
(8) The differentiable surfaces $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ and $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ are locally isometric.
(9) The surface area of a cardboard box with rectangular base and without lid does not exceed 48 square centimeters. Then, the maximal volume of such a box is 32 cubic centimeters.

Answers and comments

(1) TRUE. This follows for instance from the spectral theorem for normal operators in Linear Algebra.
(2) FALSE. Even the one-dimensional vector space $\mathbb{R}^1$ has infinitely many bases: any non-zero element is a basis.
(3) TRUE. This follows from writing 0 as a linear combination of elements in S and taking inner product against each one of the elements of S.
(4) TRUE. A standard application of the Inverse Function Theorem.
(5) FALSE. Consider for example $f : \mathbb{R}^3 \to \mathbb{R}$,

$$f(x,y,z) = \begin{cases} \frac{xyz}{x^2+y^2+z^2}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z) = (0,0,0). \end{cases}$$

It is not hard to see that $|f(x,y,z)| \leq |x|$ and therefore f is continuous at the origin. Since along the three axes f vanishes identically, all partial derivatives at the origin exist and equal 0. However, by definition one can check that f is not differentiable: approaching the limit along the line $x = y = z$ one gets the nonzero value ± 1.
(6) FALSE. The origin is a singularity.
(7) TRUE. For instance the torus of revolution in $\mathbb{R}^3$ has some points with positive curvature (the 'exterior' points), and some saddle points with negative curvature (in the 'inner' circle), thus by continuity it also has points with zero Gauss curvature.
(8) TRUE. Suitably chosen polar coordinates give such a local isometry.
(9) TRUE. Let a, b, c be the dimensions of the box, where c is its height. We want to maximize $V = abc$ subject to the constraint $S = ab + 2ac + 2bc = 48$. This can be solved using Lagrange multipliers, and one obtains that the extremal volume is attained when $a = b = 2c$.

We expect the students with the necessary background for the course to finish the quiz in well under 15 minutes, and to answer correctly at least 6 questions.