Pre-test quiz

Solutions are on the next page.

1. Find the general real solution, with maximal domain, to

$$\frac{dy}{dx} = \frac{-y}{x} + \frac{1}{x}.$$

2. Show that the linear wave equation *(i)*

$$c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0,$$

is equivalent to

$$\frac{\partial^2 u}{\partial \xi \partial \zeta} = 0,$$

where $\xi = x + ct$ and $\zeta = x - ct$.

3. Using separation of variables, find the most general solution to the linear heat diffusion equation

$$u_t = D u_{xx},$$

satisfying boundary conditions

$$u(0,t) = 0; \qquad u_x(L,t) = 0.$$

1.

$$
\begin{aligned}
\frac{dy}{dx} &= -\frac{(y-1)}{x}, \\
\frac{1}{y-1}\frac{dy}{dx} &= -\frac{1}{x}, \\
\int \frac{1}{y-1} dy &= \int -\frac{1}{x}, \\
\ln(|y-1|) &= -\ln(|x|) + c : \qquad (c \in \mathbb{R}) \\
|y-1| &= \frac{e^c}{|x|}; \qquad (c \in \mathbb{R}) \\
y-1 &= \frac{c_1}{x}, \qquad (c_1 \in \mathbb{R}) \\
y &= 1 + \frac{c_1}{x} \qquad (c_1 \in \mathbb{R}).
\end{aligned}
$$

Alternatively you can use the standard integrating factor for a first-order linear DE. The domain is $\mathbb{R}$ if $c_1 = 0$. The domain is $\mathbb{R} - \{0\}$ if $c_1 \neq 0$. In that case the maximal connected domain is either the set of positive real numbers or the set of negative real numbers.

2.

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta} = 1\frac{\partial}{\partial \xi} + 1\frac{\partial}{\partial \zeta}, \\
\frac{\partial}{\partial t} &= \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \zeta}{\partial t}\frac{\partial}{\partial \zeta} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \zeta}.
\end{aligned}
$$

$$
\begin{aligned}
& c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \\
\iff & c^2\left[\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \zeta}\right]\left[\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \zeta}\right] - \left[c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \zeta}\right]\left[c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \zeta}\right] = 0 \\
\iff & 4c^2\frac{\partial^2 u}{\partial \xi \partial \zeta} = 0 \\
\iff & \frac{\partial^2 u}{\partial \xi \partial \zeta} = 0.
\end{aligned}
$$

3.

$$
\begin{aligned}
u_t &= Du_{xx} \\
\text{Try } u &= F(x)G(t); \\
F(x)G'(t) &= DF''(x)G(t) \\
\frac{F(x)G'(t)}{FG} &= \frac{DF''(x)G(t)}{FG} \\
\implies \frac{1}{D}\frac{G'(t)}{G} &= \frac{F''(x)}{F} = \alpha \text{ (const., since x and t are independent.)}
\end{aligned}
$$

If you choose $\alpha = 0$, then the general solution for F is $F(x) = ax + b$ (general straight line function), which can't satisfy both boundary conditions unless $a = b = 0$.
If you choose $\alpha = \beta^2 > 0$, then the general solution for F is $F(x) = a\cosh(\beta x) + b\sinh(\beta x)$, which can't satisfy both boundary conditions unless $a = b = 0$.
If you choose $\alpha = -\omega^2 < 0$, the general solution for F is
$F(x) = a\cos(\omega x) + b\sin(\omega x)$.
$u(0,t) = 0 \implies F(0) = 0 \implies a = 0$, so $F(x) = b\sin(\omega x)$.
$u_x(L,t) = 0 \implies b\omega\cos(\omega L) = 0 \implies \omega L = n\pi + \frac{\pi}{2};\ n \in \mathbb{Z}$.
Hence ω must be one of the eigenvalues $\omega_n = n\pi/L + \frac{\pi}{2L};\ n \in \mathbb{Z}$.
Hence α must be one of the values $\alpha_n = -\omega_n^2$. The solution for $G(t)$ is
$G_n(t) = ce^{-D\alpha_n^2 t}$.
Since the PDE and the boundary conditions are linear, we can take linear combinations

$$u = \sum_{n=1}^{\infty} b_n \cos(\omega_n x) e^{-D\omega_n^2 t}; \quad b_n \in \mathbb{R}.$$

The series converges provided the initial condition $u(x,0)$ is bounded for $x \in [0,L]$. In that case we can construct the Fourier coefficients b_n of $u(x,0)$.