Finite Element Method: Background Materials

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July 5, 2019

Field

We define a **field** $\mathbb{F}$ as a set together with two operations, usually called addition and multiplication, and denoted by $+$ and $\cdot$, respectively, such that the following axioms hold.

1. Closure of $\mathbb{F}$ under addition and multiplication: $a+b \in \mathbb{F}$, and $a \cdot b \in \mathbb{F}$ for all $a, b \in \mathbb{F}$.
2. Associativity of addition and multiplication: $a+(b+c)=(a+b)+c$, and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{F}$.
3. Commutativity of addition and multiplication: $a+b=b+a$, and $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{F}$.
4. Existence of additive and multiplicative identity elements: There exists some element in $\mathbb{F}$ and denoted by 0, such that for all $a \in \mathbb{F}$, $a+0=a$. Similarly, there is some element in $\mathbb{F}$ and denoted by 1, such that for all $a \in \mathbb{F}$, $a \cdot 1 = a$.
5. Existence of additive inverses and multiplicative inverses: For every $a \in \mathbb{F}$, there exists an element $-a \in \mathbb{F}$, such that $a+(-a)=0$. Likewise, for any $a \in \mathbb{F}$ other than 0, there exists an element $a^{-1} \in \mathbb{F}$, such that $a \cdot a^{-1} = 1$.
6. Distributivity of multiplication over addition: $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in \mathbb{F}$.

Vector Space

A *vector space* over a field $\mathbb{F}$ is a set $\mathcal{V}$ together with two operations vector addition, denoted $v + w \in \mathcal{V}$ for $v, w \in \mathcal{V}$ and scalar multiplication, denoted $av \in \mathcal{V}$ for $a \in \mathbb{F}$ and $v \in \mathcal{V}$, such that the following assumptions are satisfied:

1. $v + w = w + v, \ v, w \in \mathcal{V}$.
2. $u + (v + w) = (u + v) + w, \ u, v, w \in \mathcal{V}$.
3. There exists an element $0 \in \mathcal{V}$, called the zero vector, such that $v + 0 = v, \ v \in \mathcal{V}$.
4. There exists an element $\tilde{v} \in \mathcal{V}$, called the additive inverse of v, such that $v + \tilde{v} = 0$, $v \in \mathcal{V}$.
5. $a(bv) = (ab)v, \ a, b \in \mathbb{F}$ and $v \in \mathcal{V}$.
6. $a(v + w) = av + aw, \ a \in \mathbb{F}$ and $v, w \in \mathcal{V}$.
7. $(a + b)v = av + bv, \ a, b \in \mathbb{F}$ and $v \in \mathcal{V}$.
8. There exists a multiplicative identity $1 \in \mathbb{F}$ such that $1v = v, \ v \in \mathcal{V}$.

Vector Space

The elements of a vector space are called *vectors*. A subset $\mathcal{S}$ of a vector space $\mathcal{V}$ is a *subspace* of $\mathcal{V}$ if it is a vector space with respect to the vector space operations on $\mathcal{V}$. A subspace which is a proper subset of the whole space is called a *proper subspace*.

If $v_1, v_2, \ldots, v_n$ are some elements of a vector space $\mathcal{V}$, by a *linear combination* of $v_1, \ldots, v_n$ we mean an element in $\mathcal{V}$ of the form $a_1v_1 + \cdots + a_nv_n$, with $a_i \in \mathbb{F}, i = 1, \ldots, n$.

Let S be a subset of $\mathcal{V}$. The set of all *linear combinations* of elements of S is called the *span* of S and is denoted by $\operatorname{span} S$. If $S = \{v_1, v_2, \ldots, v_n\}$, then we often write $S = \{v_i\}_{i=1}^n$.

A subset $S = \{v_i\}_{i=1}^n$ of $\mathcal{V}$ is said to be *linearly independent* if and only if

$$a_1v_1 + \cdots + a_nv_n = 0, \quad \Longrightarrow \quad a_i = 0, i = 1, \ldots, n.$$

A subset is said to be *linearly dependent* if it is not linearly independent.

A Basis of a Vector Space

S is said to be a *basis* of $\mathcal{V}$ if it is linearly independent and span $S = \mathcal{V}$. The *dimension* of a finite dimensional vector space $\mathcal{V}$ is the number of elements in a basis for $\mathcal{V}$. The number of elements in a set is termed the *cardinality* of the set.

Let $\{v_i\}_{i=1}^n$ be a basis for $\mathcal{V}$. For $v = a_1 v_1 + \cdots, a_n v_n$, $\{a_i\}_{i=1}^n$ are called coordinates of v with respect to the basis $\mathcal{V}$.

- An example of a vector space is the *Euclidean space* $\mathbb{R}^n$ over the field of real or complex numbers.
- Another example is the the *space of polynomials* of degree $m \in \mathbb{N}$ on $\mathbb{R}$ by

$$\mathcal{P}_m(\mathbb{R}) = \left\{ p : p(x) = \sum_{i=0}^{m} a_i x^i, \ x \in \mathbb{R} \right\}.$$

  This is a vector space over the field of real or complex numbers.

A Norm on a Vector Space

A *norm* $\|\cdot\|$ on a vector space $\mathcal{V}$ is a function from $\mathcal{V}$ to $\mathbb{R}$ such that for every $v, w \in \mathcal{V}$ and $a \in \mathbb{F}$ the following three properties are fulfilled

1. $\|v\| \geq 0$, and $\|v\| = 0 \Longleftrightarrow v = 0$.
2. $\|av\| = |a|\|v\|$.
3. $\|v + w\| \leq \|v\| + \|w\|$.

A vector space $\mathcal{V}$ together with a norm is called a *normed vector space* or simply a *normed space*. An *inner product space* $\mathcal{V}$ is also a *normed space*.

The *Euclidean space* $\mathbb{R}^n$ is a *normed space*, where the norm of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is given by

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

Inner Product on a Vector Space

An *inner product* on a vector space $\mathcal{V}$ is a map from $\mathcal{V}$ to $\mathbb{F}$ which satisfies the following assumptions

1. $\langle v, v\rangle \geq 0,\ v \in \mathcal{V}$, and $\langle v, v\rangle = 0 \Longleftrightarrow v = 0$.
2. $\langle v + w, z\rangle = \langle v, z\rangle + \langle w, z\rangle,\ \ v, w, z \in \mathcal{V}$.
3. $\langle v, az\rangle = a\langle v, z\rangle,\ v, z \in \mathcal{V}$ and $a \in \mathbb{F}$.
4. $\langle v, w\rangle = \langle w, v\rangle,\ v, w \in \mathcal{V}$.

A vector space $\mathcal{V}$ together with an inner product $\langle\cdot,\cdot\rangle$ is called an *inner product space*.

Two vectors v and w in an inner product space are said to be *orthogonal* if $\langle v, w\rangle = 0$. Two vectors v and w are said to be *orthonormal* if they are *orthogonal* and $||v|| = ||w|| = 1$. The *Euclidean space* $\mathbb{R}^n$ is an inner product space with inner product defined by $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^{n} x_i y_i$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. The norm $\|\mathbf{x}\|$ is induced by the inner product $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x}\rangle^{\frac{1}{2}}$.

Function Space of Continuous Functions

Let $\mathbb{Z}_+$ be the set of non-negative integers, and $d \in \{1, 2, 3\}$. In the following Ω is an open subset of $\mathbb{R}^d$ with piecewise smooth boundary.

1. The short-hand notation for the mixed partial derivative of a function can be written in terms of the so-called multi-index notation α, which is an d-tuple of non-negative integers α_i so that $\alpha := (\alpha_1, \cdots, \alpha_d)$. The length of α is given by $|\alpha| := \sum_{i=1}^{d} \alpha_i$. We use $\mathbf{0} = (0, \cdots, 0)$.
2. $C^k(\Omega)$ denotes the set of all continuous real-valued functions on Ω such that $D^\alpha u$ is also continuous on Ω for all
$$\alpha = (\alpha_1, \cdots, \alpha_d) \quad \text{with} \quad |\alpha| \le k, \quad k \in \mathbb{Z}_+.$$
3. For a function $\phi \in C^{|\alpha|}(\Omega)$, $D^\alpha \phi$ will denote the usual point-wise mixed partial derivative $\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \phi$.

Function Space

Suppose that $\Omega \subset \mathbb{R}^3$. Then for a function $u \in C^3(\Omega)$ we have

$$\sum_{|\alpha|=3} D^\alpha u = \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} + \frac{\partial^3 u}{\partial x_3^3} + \frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_1^2 \partial x_3} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2}$$
$$+ \frac{\partial^3 u}{\partial x_1 \partial x_3^2} + \frac{\partial^3 u}{\partial x_1^2 \partial x_3} + \frac{\partial^3 u}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3}.$$

If Ω is a bounded open set, $C^k(\bar{\Omega})$ denotes the set of all $u \in C^k(\Omega)$ such that $D^\alpha u$ can be extended to a continuous function on $\bar{\Omega}$, the closure of Ω, for all

$$\alpha = (\alpha_1, \cdots, \alpha_d) \quad \text{with} \quad |\alpha| \leq k.$$

We can use the following norm for functions in $C^k(\bar{\Omega})$:

$$\|u\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \leq k} \sup |D^\alpha u(\boldsymbol{x})|.$$

It is standard to write $C(\bar{\Omega})$ for $C^0(\bar{\Omega})$ when $k = 0$.

Function Space: Example

Let $w : \mathbb{R}^d \to \mathbb{R}$ be defined by

$$w(\boldsymbol{x}) = \begin{cases} e^{-\frac{1}{1-\|\boldsymbol{x}\|^2}}, & \text{if} \quad |\boldsymbol{x}| < 1 \\ 0, & \text{otherwise;} \end{cases}$$

where $\|\boldsymbol{x}\| = \sqrt{\sum_{k=1}^d x_k^2}$. Note that the *support* of a continuous function defined on an open set $\Omega \subset \mathbb{R}^d$ is the closure in Ω of the set

$$\{\boldsymbol{x} \in \Omega : u(\boldsymbol{x}) \neq 0\}$$

denoted by $\operatorname{supp} u$. If $\operatorname{supp} u$ is a bounded subset of Ω, u is said to be compactly supported in Ω. *A closed and bounded set $K \subset \mathbb{R}^d$ is compact.* In the previous example the support of w is compact with

$$\operatorname{supp} w = \{\boldsymbol{x} \in \mathbb{R}^d : \ \|\boldsymbol{x}\| \leq 1\}.$$

Let $C_0^k(\Omega)$ be the set of all $u \in C^k(\Omega)$ such that $\operatorname{supp} u$ is a bounded subset of Ω. We often use the notation

$$C_0^\infty(\Omega) = \bigcap_{k=0}^{\infty} C_0^k(\Omega).$$

Weak Derivative

The set of functions which are differentiable infinitely many times is denoted by $C^\infty(\Omega)$ and the set $L^1_{loc}(\Omega)$ of locally integrable functions is defined by

$$L^1_{loc}(\Omega) := \{u|\, u \in L^1(K), \text{ compact } K \subset \Omega\}.$$

The definition of the weak derivative is given in terms of the set of functions $\mathcal{D}(\Omega)$ with $\mathcal{D}(\Omega) := \{u \in C^\infty(\Omega) : \operatorname{supp} u \text{ is compact}\}$. A function $f \in L^1_{loc}(\Omega)$ has a weak derivative $D^\alpha_w f$ if there exists a function $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega g(\boldsymbol{x})\phi(\boldsymbol{x})\, dx = (-1)^{|\alpha|} \int_\Omega f(\boldsymbol{x}) D^\alpha_w \phi(\boldsymbol{x})\, dx, \quad \phi \in \mathcal{D}(\Omega).$$

If such a g exists, we define $D^\alpha_w f := g$. From now on, we will merely use $D^\alpha \phi$ instead of $D^\alpha_w \phi$ to denote the weak derivative of ϕ.

Weak Derivative: Example

Consider the function $u(x) = (1 - |x|)_+$, where

$$(x)_+ = \begin{cases} 0, & \text{if} \quad x < 0 \\ x, & \text{else.} \end{cases}$$

This function is not differentiable at the points 0 and ± 1. However, we can compute a weak derivative of u. For $v \in \mathcal{D}(\mathbb{R})$

$$\begin{aligned} \int_{\mathbb{R}} u(x) v'(x)\, dx &= \int_{-1}^{1} (1 - |x|) v'(x)\, dx \\ &= \int_{-1}^{0} (1 + x) v'(x)\, dx + \int_{0}^{1} (1 - x) v'(x)\, dx \\ &= \int_{-1}^{0} (-1) v(x)\, dx + \int_{0}^{1} (+1) v(x)\, dx = - \int_{\mathbb{R}} g(x) v(x)\, dx, \end{aligned}$$

where

$$g(x) = \begin{cases} 0, & \text{if } x < -1 \\ 1, & \text{if } x \in (-1, 0) \\ -1, & \text{if } x \in (0, 1) \\ 0, & \text{if } x > 1. \end{cases}$$

Hence $g = Du$.

Normed and Inner Product Spaces

The function space $C^k(\Omega)$ equipped with the the following norm

$$\|u\|_{C^k(\Omega)} = \sup_{\boldsymbol{x}\in\Omega} \sum_{|\alpha|\leq k} |D^\alpha u(\boldsymbol{x})|$$

is an example of normed space. We can measure the distance between two functions u and v of $C^k(\Omega)$ by using the above norm.

Let H be a normed space equipped with the norm $\|\cdot\|_H$. A sequence $\{x_n\}$ in a normed space H converges to an element $x \in H$ if $\lim_{n\to\infty} \|x_n - x\|_H \to 0$. We write $x_n \to x$. This is convergence in the norm.

If a function space H is equipped with an inner product $\langle \cdot, \cdot \rangle_H$, it is called an inner product space. For example, the set of square integrable functions $L^2(\Omega)$ is an inner product space. The inner product for two functions f and g of $L^2(\Omega)$ is defined as

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f g \, dx.$$

This inner product also induces the following norm on $L^2(\Omega)$:

$$\|f\|_{L^2(\Omega)} = \sqrt{\int_\Omega f^2 \, dx}.$$