Introduction to Model Theory for C*-Algebras

Pre-enrollment quiz

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1. (Book of Proof, Exercise 2.5.11) Let P, Q, R, S be mathematical statements. Suppose P is false and the statement
$$(R \implies S) \iff (P \wedge Q)$$
is true. Find the truth values of R and S. (Is R true or false? Is S true or false?)

2. (Book of Proof, Exercise 1.8.13) Let I, J be index sets with $J \neq \emptyset$ and $J \subseteq I$. Also let $\{A_\alpha\}_{\alpha \in I}$ be a collection of sets indexed by I.
Then is it true that $\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$? What about $\bigcap_{\alpha \in J} A_\alpha \subseteq \bigcap_{\alpha \in I} A_\alpha$?

3. The *center* of a ring A is the set of all elements that commute with everything in the ring, denoted $Z(A)$:
$$Z(A) = \{x \in A : \, xy = yx \, \forall y \in A\}.$$
Show that the center of $M_n(\mathbb{C})$ is equal to just scalar multiples of the identity.

4. Let $A \subseteq \mathbb{R}$ be bounded and infinite. Prove that A has an accumulation point; that is, there exists $x \in \mathbb{R}$ such that every open interval containing x contains a point of $A \setminus x$.
(Hint: use the Bolzano–Weierstrass theorem.)

5. This question is a precursor to the Hahn–Banach (extension) theorem, which is central to functional analysis. (You are not expected to know or use Hahn-Banach to solve this question!)

   Let V be a real vector space, and let $p\colon V \to \mathbb{R}$ be a sublinear function. (This means for all $\mathbf{x}, \mathbf{y} \in V$ and all $\lambda \geq 0$, $p(\mathbf{x} + \mathbf{y}) \leq p(\mathbf{x}) + p(\mathbf{y})$ and $p(\lambda \mathbf{x}) = \lambda p(\mathbf{x})$.)

   Let $M \subseteq V$ be a linear subspace, and suppose that $f\colon M \to \mathbb{R}$ is linear and satisfies
$$f(\mathbf{m}) \leq p(\mathbf{m}) \qquad \text{for every } \mathbf{m} \in M.$$
   Choose $\mathbf{v} \in V \setminus M$, and define
$$M_1 = M + \operatorname{span}\{\mathbf{v}\}.$$

   (a) Show that every linear extension $F\colon M_1 \to \mathbb{R}$ of f must have the form
$$F(\mathbf{m} + t\mathbf{v}) = f(\mathbf{m}) + tc$$
   for some constant $c \in \mathbb{R}$.

   (b) Show that the inequality $F(\mathbf{x}) \leq p(\mathbf{x})$ for all $\mathbf{x} \in M_1$ holds if and only if
$$f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v}) \leq c \leq p(\mathbf{m} + \mathbf{v}) - f(\mathbf{m}) \qquad \text{for every } \mathbf{m} \in M,$$
   where c is as in Part 5a.

(c) Prove that

$$\sup_{\mathbf{m}\in M}\big(f(\mathbf{m}) - p(\mathbf{m}-\mathbf{v})\big) \le \inf_{\mathbf{m}\in M}\big(p(\mathbf{m}+\mathbf{v}) - f(\mathbf{m})\big).$$

(d) Conclude that there exists at least one value of $c \in \mathbb{R}$ for which the corresponding function F is a linear extension of f to M_1 satisfying

$$F(\mathbf{x}) \le p(\mathbf{x}) \qquad \text{for every } \mathbf{x} \in M_1.$$

(e) Apply the result to the following concrete example: $V = \mathbb{R}^2$, $M = \{(x,0) : x \in \mathbb{R}\}$, with $f(x,0) = x$ and

$$p(x,y) = |x| + 2|y|.$$

Determine all linear functionals $F\colon \mathbb{R}^2 \to \mathbb{R}$ that extend f and satisfy

$$F(x,y) \le p(x,y) \qquad \text{for every } (x,y) \in \mathbb{R}^2.$$