Introduction to Model Theory for C*-Algebras

Pre-enrollment quiz w/ solutions

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1. (Book of Proof, Exercise 2.5.11) Let P, Q, R, S be mathematical statements. Suppose P is false and the statement
$$(R \implies S) \iff (P \wedge Q)$$
is true. Find the truth values of R and S. (Is R true or false? Is S true or false?)

   Solution: Since P is false, we know $P \wedge Q$ is false. Thus, $R \implies S$ is false as well, and this holds only if R is true and S is false.

2. (Book of Proof, Exercise 1.8.13) Let I, J be index sets with $J \neq \emptyset$ and $J \subseteq I$. Also let $\{A_\alpha\}_{\alpha \in I}$ be a collection of sets indexed by I.
Then is it true that $\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$? What about $\bigcap_{\alpha \in J} A_\alpha \subseteq \bigcap_{\alpha \in I} A_\alpha$?

   Solution: The first containment holds: suppose $x \in \bigcup_{\alpha \in J} A_\alpha$. Then there exists some index $j \in J$ so that $x \in A_j$. Since $J \subseteq I$, it follows that $j \in I$, and so $x \in \bigcup_{\alpha \in I} A_\alpha$.
The second containment is false. As a counterexample, consider the sets $A_1 = \{a, b\}$ and $A_2 = \{a\}$, as well as $J = \{1\}$ and $I = \{1, 2\}$. Then $\bigcap_{\alpha \in J} A_\alpha = A_1 = \{a, b\}$, but $\bigcap_{\alpha \in I} A_\alpha = A_1 \cap A_2 = \{a\}$.

3. The *center* of a ring A is the set of all elements that commute with everything in the ring, denoted $Z(A)$:
$$Z(A) = \{x \in A : \; xy = yx \, \forall y \in A\}.$$
Show that the center of $M_n(\mathbb{C})$ is equal to just scalar multiples of the identity.

   Solution: One containment is easy: it is clear that for any scalar multiple of the identity $B = \lambda I$, we have $AB = BA$ for all matrices $A \in M_n(\mathbb{C})$.

   Now let C be any matrix in the center of $M_n(\mathbb{C})$. Let E_{ij} denote the elementary matrix with 1 in the (i, j)-th entry and 0's elsewhere. Then in particular, computing $CE_{i,j} = E_{i,j}C$ shows that C must be a scalar multiple of the identity.

   Explicitly, CE_{ij} is the matrix with the j-th column equal to the i-th column of C and 0's elsewhere, and $E_{ij}C$ is the matrix with the i-th row equal to the j-th row of C and 0's elsewhere.
$$(CE_{ij})_{k\ell} = C_{ki}\delta_{\ell=j} \tag{1}$$
$$(E_{ij}C)_{k\ell} = C_{j\ell}\delta_{k=i} \tag{2}$$
   Equating these two means for all indices i, j, k, ℓ, we have
$$C_{ki}\delta_{\ell=j} = C_{j\ell}\delta_{k=i},$$
   so both sides are zero unless $\ell = j$ and $k = i$, in which case $C_{ii} = C_{jj}$. We conclude that $C = C_{11} \cdot I$ is a scalar multiple of the identity.

4. Let $A \subseteq \mathbb{R}$ be bounded and infinite. Prove that A has an accumulation point; that is, there exists $x \in \mathbb{R}$ such that every open interval containing x contains a point of $A \setminus x$.
(Hint: use the Bolzano–Weierstrass theorem.)

Solution: Assume the conclusion is false, that is: For any $x \in \mathbb{R}$, there exists $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \cap (A \setminus \{x\}) = \emptyset$.

Since A is infinite, it contains a sequence $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \neq x_m$ for $n \neq m$. Since A is bounded, so is the sequence $\{x_n\}_n$, and so by Bolzano–Weierstrass and by passing to a subsequence, we can without loss of generality assume that $x_n \to x$ for some $x \in \mathbb{R}$.

Let $\epsilon := \epsilon_x > 0$ be the positive number as specified in the first paragraph for this particular x. Since $(x - \epsilon, x + \epsilon)$ is an open neighborhood of x in $\mathbb{R}$ and since $\lim_n x_n = x$, there exists $N \in \mathbb{N}$ such that $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \geq N$. Since $x_n \in A$ and since $(x - \epsilon, x + \epsilon) \cap (A \setminus \{x\}) = \emptyset$, this implies that $x_n = x$ for all $n \geq N$, which contradicts our assumption that $x_n \neq x_m$ for $n \neq m$.

5. This question is a precursor to the Hahn–Banach (extension) theorem, which is central to functional analysis. (You are not expected to know or use Hahn-Banach to solve this question!)

Let V be a real vector space, and let $p\colon V \to \mathbb{R}$ be a sublinear function. (This means for all $\mathbf{x}, \mathbf{y} \in V$ and all $\lambda \geq 0$, $p(\mathbf{x} + \mathbf{y}) \leq p(\mathbf{x}) + p(\mathbf{y})$ and $p(\lambda \mathbf{x}) = \lambda p(\mathbf{x})$.)

Let $M \subseteq V$ be a linear subspace, and suppose that $f\colon M \to \mathbb{R}$ is linear and satisfies

$$f(\mathbf{m}) \leq p(\mathbf{m}) \qquad \text{for every } \mathbf{m} \in M.$$

Choose $\mathbf{v} \in V \setminus M$, and define

$$M_1 = M + \operatorname{span}\{\mathbf{v}\}.$$

(a) Show that every linear extension $F\colon M_1 \to \mathbb{R}$ of f must have the form

$$F(\mathbf{m} + t\mathbf{v}) = f(\mathbf{m}) + tc$$

for some constant $c \in \mathbb{R}$.

Solution: An arbitrary element of M_1 is of the form $\mathbf{m} + t\mathbf{v}$ for $\mathbf{m} \in M$ and $t \in \mathbb{R}$. Since F is linear, we have $F(\mathbf{m} + t\mathbf{v}) = F(\mathbf{m}) + tF(\mathbf{v})$, and since F extends f, this is equal to $f(\mathbf{m}) + tF(\mathbf{v})$. We can now let $c := F(\mathbf{v})$.

(b) Show that the inequality $F(\mathbf{x}) \leq p(\mathbf{x})$ for all $\mathbf{x} \in M_1$ holds if and only if

$$f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v}) \leq c \leq p(\mathbf{m} + \mathbf{v}) - f(\mathbf{m}) \qquad \text{for every } \mathbf{m} \in M,$$

where c is as in Part 5a.

Solution: Assume first that $F(\mathbf{x}) \leq p(\mathbf{x})$ for all $\mathbf{x} \in M_1$ and fix $\mathbf{m} \in M$. By assumption, we have $f(\mathbf{m}) \pm c = F(\mathbf{m} \pm \mathbf{v}) \leq p(\mathbf{m} \pm \mathbf{v})$; rearranging yields $f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v}) \leq c$ (for $-$) and $c \leq p(\mathbf{m} + \mathbf{v}) - f(\mathbf{m})$ (for $+$).

Next assume that the displayed equation holds for all elements of M; we will refer to this as (†). Fix $\mathbf{x} = \mathbf{m} + t\mathbf{v} \in M_1$. We do a case distinction:

$t = 0$: This is by assumption on f: $F(\mathbf{x}) = f(\mathbf{x}) \leq p(\mathbf{x})$.

$t > 0$: By Part 5a and by the right-hand inequality of (†) applied to $\frac{1}{t}\mathbf{m} \in M$, we have

$$F(\mathbf{x}) = f(\mathbf{m}) + tc \overset{(\dagger)}{\leq} f(\mathbf{m}) + t\left(p(\tfrac{1}{t}\mathbf{m} + \mathbf{v}) - f(\tfrac{1}{t}\mathbf{m})\right) = t\,p(\tfrac{1}{t}\mathbf{m} + \mathbf{v}) = p(\mathbf{m} + t\mathbf{v}).$$

$t < 0$: Write $\mathbf{m} = -t(\frac{1}{-t}\mathbf{m})$. By Part 5a and by the left-hand inequality of (†) applied to $\frac{1}{-t}\mathbf{m} \in M$, we have

$$F(\mathbf{x}) = -t\,f(\tfrac{1}{-t}\mathbf{m}) + tc \overset{(\dagger)}{\leq} -t\left(c + p(\tfrac{1}{-t}\mathbf{m} - \mathbf{v})\right) + tc = -t\,p(\tfrac{1}{-t}\mathbf{m} - \mathbf{v}) = p(\mathbf{m} + t\mathbf{v}).$$

(c) Prove that

$$\sup_{\mathbf{m}\in M}\big(f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v})\big) \leq \inf_{\mathbf{m}\in M}\big(p(\mathbf{m} + \mathbf{v}) - f(\mathbf{m})\big).$$

Solution: It suffices to show that for any $\mathbf{m}, \mathbf{n} \in M$, we have

$$f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v}) \leq p(\mathbf{n} + \mathbf{v}) - f(\mathbf{n}).$$

Rearranging, this is equivalent to showing

$$f(\mathbf{m} + \mathbf{n}) = f(\mathbf{m}) + f(\mathbf{n}) \leq p(\mathbf{n} + \mathbf{v}) + p(\mathbf{m} - \mathbf{v}).$$

By assumption on f, we have $f(\mathbf{m} + \mathbf{n}) \leq p(\mathbf{m} + \mathbf{n})$. By assumption on p, we have

$$p(\mathbf{m} + \mathbf{n}) = p\left((\mathbf{m} - \mathbf{v}) + (\mathbf{n} + \mathbf{v})\right) \leq p(\mathbf{m} - \mathbf{v}) + p(\mathbf{n} + \mathbf{v}),$$

which finishes the claim.

(d) Conclude that there exists at least one value of $c \in \mathbb{R}$ for which the corresponding function F is a linear extension of f to M_1 satisfying

$$F(\mathbf{x}) \leq p(\mathbf{x}) \qquad \text{for every } \mathbf{x} \in M_1.$$

Solution: By Part 5c, we are able to fix a number $c \in \mathbb{R}$ such that

$$\sup_{\mathbf{m}\in M}\big(f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v})\big) \leq c \leq \inf_{\mathbf{m}\in M}\big(p(\mathbf{m} + \mathbf{v}) - f(\mathbf{m})\big).$$

In particular, we have for all $\mathbf{m} \in M$ that

$$f(\mathbf{m}) - p(\mathbf{m} - \mathbf{v}) \leq c \leq p(\mathbf{m} + \mathbf{v}) - f(\mathbf{m}).$$

By (a converse of) Part 5a, we can define $F(\mathbf{m} + t\mathbf{v}) := f(\mathbf{m}) + tc$ for this particular c; this defines a linear map F that extends f to M_1. By Part 5b and the displayed equation above, this F satisfies $F(\mathbf{x}) \leq p(\mathbf{x})$ for all $\mathbf{x} \in M_1$.

(e) Apply the result to the following concrete example: $V = \mathbb{R}^2$, $M = \{(x, 0) : x \in \mathbb{R}\}$, with $f(x, 0) = x$ and

$$p(x, y) = |x| + 2|y|.$$

Determine all linear functionals $F\colon \mathbb{R}^2 \to \mathbb{R}$ that extend f and satisfy

$$F(x, y) \leq p(x, y) \qquad \text{for every } (x, y) \in \mathbb{R}^2.$$

Solution: Here, our preferred choice of $\vec{v} \in V \setminus M$ is $\mathbf{v} := (0,1)$. From (a converse of) Part 5a, we know that any extension of f must be of the form $F_c(x,y) = x + yc$ for some fixed $c \in \mathbb{R}$. We know from a combination of Parts 5b and 5c that F_c satisfies $F_c(x,y) \leq p(x,y)$ for all $(x,y) \in \mathbb{R}^2$ if and only if

$$\sup_{x \in \mathbb{R}} \big(x - (|x| + 2)\big) \leq \inf_{x \in \mathbb{R}} \big(|x| + 2 - x\big).$$

Note that the left-hand side equals -2 and the right-hand side equals 2. This means that the only linear extensions F_c of f that satisfy $F(\mathbf{x}) \leq p(\mathbf{x})$ are those corresponding to $c \in [-1, 1]$.