Mathematics of extreme events in nature
Pre-enrollment quiz

1. a) Find $y = \ln(p_1/p_2)$ when $p_1 = 1000$ hPa and $p_2 = 500$ hPa.
   b) The ideal gas law is $p = \rho RT$. Rearrange this equation to make density (ρ) the subject.

2. Let $P = (0, -2, 2)$ and $Q = (2, 0, 1)$ be points in three-dimensional space. Which of these points is closest to the origin?

3. The hydrostatic equation is given by $dp/dz = -\rho g$.
   a) Rearrange this equation to make dz the subject.
   b) If pressure is measured in Pascals (Pa) and height in metres (m), what are the units of dp/dz?
   c) Suppose $dp/dz < 0$. Which statement is correct?
   - i Pressure increases with height.
   - ii Pressure decreases with height.
   - iii Pressure remains constant.

4. a) Differentiate $\ln(x^2 + 1)$.
   b) Evaluate the definite integral $\int_{500}^{1000} \frac{1}{p}\, dp$.

5. a) Given $\Phi = \int_0^z g\, dz$ calculate the differential form $d\Phi$.
   b) Compute the differential of $z = \cos(xy)$.

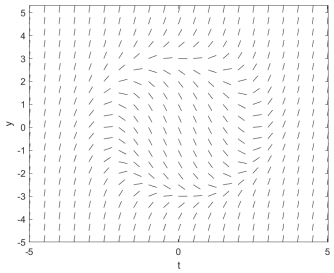
6. Let $f(x, y) = \sin(xy)$.
   a) Calculate $\frac{\partial^2 f}{\partial y\, \partial x}$.
   b) Calculate f_{yy}.

7. Find the normal vector of the surface $ax + by + cz = d$ at any point.

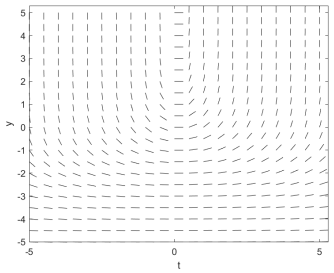
8. Let $f(x, y) = \frac{x+y}{1+x^2}$.
   a) Calculate $\mathrm{grad} f(1, -2)$.
   b) Let $\hat{v}$ be a unit vector in the direction $3\hat{\imath} - 2\hat{\jmath}$. Calculate $\nabla f(1, -2) \cdot \hat{v}$.

9. The collection of small lines in the figures below constitute a slope field for a given ODE and the slope of each line satisfies $\dot{y} = dy/dt$ for (t, y) on a regular grid. Match the following differential equations with their slope field.

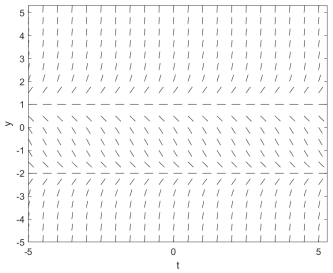
   (i) $\dot{y} = y^2/3 + t^2/2 - 3$ (ii) $\dot{y} = (y - 1)(y + 2)$ (iii) $\dot{y} = te^y$



(a)



(b)



(c)

10. Solve the following Initial Value Problem (where $y' = \frac{dy}{dx}$),

$$y'' - 2y' - 3y = 0 \text{ given } y(0) = 2 \text{ and } y'(0) = 14.$$

11. Calculate $\frac{\partial(x,y)}{\partial(r,\theta)}$ for $x = r\cos\theta$ and $y = r\sin\theta$.

12. Calculate the eigenvalues of $\mathbf{A}$ when $\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & \cos\theta \end{bmatrix}$.

Solution 1.
a) $\ln(2)$
b) $\rho = p/(RT)$

Solution 2.
Q

Solution 3.
a) $dz = (1/(\rho g))\, dp$
b) Pa m^{-1}
c) ii) Pressure decreases with height

Solution 4.
a) $\frac{2x}{x^2+1}$
b) $\ln(1000/500) = \ln(2) \approx 0.693$

Solution 5.
a) $d\Phi = g\ dz$
b) $dz = -y\sin(xy)\, dx - x\sin(xy)\, dy$

Solution 6.
a) $\cos(xy) - xy\, sin(xy)$
b) $-x^2\sin(xy)$

Solution 7.
$[a, b, c]$

Solution 8.
a) $\hat{\imath} + \frac{1}{2}\hat{\jmath}$
b) $\frac{2}{\sqrt{13}}$

Solution 9.
a) (i)
b) (iii)
c) (ii)

Solution 10.
$y = 4e^{3x} - 2e^{-x}$

Solution 11.
r

Solution 12.
$\lambda_{1,2} = \cos\theta$