Warm-up questions

Problem 1. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let ν be the outward unit-vector normal to the boundary $\partial\Omega$. Suppose that u and v are two functions in $C^2(\bar{\Omega})$ (that is, u and v have continuous second-order partial derivatives on $\Omega \cup \partial\Omega$).

1. Prove Green's first identity:

$$\int_\Omega v\Delta u \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx \; = \; \int_{\partial\Omega} v \, \partial_\nu u \, dS \, .$$

   In this expression, $\nabla u = (\partial_{x^1} u, \ldots, \partial_{x^n} u)$ is the **gradient**, $\Delta u = \sum_{i=1}^n \partial^2_{x^i x^j} u$ is the **Laplacian operator**, and $\partial_\nu u = \nu \cdot \nabla u$ is the directional derivative in the direction of ν. *Hint:* $\Delta = div\nabla$, *where for a vector-field* $F = (F_1, \ldots, F_2)$, *the* **divergence operator** *is* $div\, F = \sum_{i=1}^n \partial_{x^j} F_j$. *Use the divergence theorem.*

2. Prove Green's second identity:

$$\int_\Omega v\Delta u \, dx - \int_\Omega u\Delta v \, dx \; = \; \int_{\partial\Omega} \big(v \, \partial_\nu u - u \, \partial_\nu v \big) \, dS \, .$$

Problem 2. Let $B_r(x)$ be the open ball in $\mathbb{R}^n$ of center x and radius r. Let u be continuous on $B_1(x)$. Prove that

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) dy \; = \; u(x) \, ,$$

where $|B_r(x)| = \int_{B_r(x)} dy$ denotes the volume of the ball.

Problem 3. Let $I \subset \mathbb{R}$ be an interval and let $f \in C^0(\bar{I})$. For $\alpha \geq 0$, define

$$[f]_{C^{0,\alpha}(\bar{I})} \; := \; \sup_{x,y \in \bar{I} \, ; \, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \, .$$

If this quantity is finite, we say that f is **Hölder continuous of order** α. Suppose that f is differentiable on I and Hölder continuous of order $\alpha > 1$. Prove that f must be constant. *Hint: Mean value theorem.*

Problem 4.

1. Let $p \in (1, \infty)$, $a \geq 0$, and $b \geq 0$ be three real numbers. Prove that

$$a^{1/p} b^{1-1/p} \; \leq \; a/p + (1 - 1/p) b \, .$$

   Hint: Apply logarithm to both sides. Use the fact that log is **concave**.

2. Put $q = 1 - 1/p$. Use your answer to the previous question to show that if $A \geq 0$ and $B \geq 0$, then **Young's inequality** holds:

$$AB \; \leq \; \frac{A^p}{p} + \frac{B^q}{q} \, .$$

   Hint: Put $A = a^{1/p}$. *Choose* B *suitably.*

Problem 5. Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that the function $u \in C^2(\Omega)$ is **harmonic**:

$$\Delta u \;=\; 0 \qquad \text{on} \;\; \Omega.$$

Let $B_R(y) \subset\subset \Omega$ (which means that the ball is **compactly contained** in Ω, that is $\overline{B_R(y)} \subset \Omega$.).

1. Let $\rho \in (0, R)$. Prove

$$\int_{B_\rho(y)} \partial_\nu u \, dS \;=\; 0 \, .$$

2. Introduce radial/angular coordinates (r, ω) on $B_R(y)$ such that $r = |x - y|$ and $\omega = (x - y)/r$. Write $u(x) = u(y + r\omega)$. Using 1., obtain the following steps:

$$\begin{aligned} 0 \;&=\; \int_{\partial B_\rho(y)} \frac{\partial u}{\partial r}(y + \rho\omega) \, dS \;=\; \rho^{n-1} \int_{|\omega|=1} \frac{\partial u}{\partial r}(y + \rho\omega) \, d\omega \;=\; \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|\omega|=1} u(y + \rho\omega) \, d\omega \\ &=\; \rho^{n-1} \frac{\partial}{\partial \rho} \left(\rho^{1-n} \int_{\partial B_\rho(y)} u \, dS \right) . \end{aligned}$$

3. Deduce from the previous question that

$$\rho^{1-n} \int_{\partial B_\rho(y)} u \, dS \;=\; R^{1-n} \int_{\partial B_R(y)} u \, dS \qquad \forall \; \rho \in (0, R) \, .$$

4. Use Problem 2. to deduce that

$$u(y) \;=\; \frac{1}{|\partial B_R(y)|} \int_{\partial B_R(y)} u \, dS \qquad \forall \; \rho \in (0, R) \, .$$

   This is called **mean value property for harmonic functions.** It is a fundamental fact; you should halt and spend a few minutes to appreciate it :). *Hint:* $|B_R(y)|/|B_\rho(y)| = (R/\rho)^{n-1}$.

5. Prove that $\Delta u^2 \geq 0$. We say that u^2 is **subharmonic**.

6. State and prove an analogue of the mean value property for subharmonic functions.

Solutions

Problem 1.

1. This is a task for the **divergence theorem**:

$$\int_\Omega div\, F\, dx \;=\; \int_{\partial\Omega} \nu \cdot F\, dS \qquad \text{for all vector-fields } F.$$

   Now choose $F = u\nabla v$. Note that

$$div\, F \;=\; \sum_{j=1}^n \partial_{x^j}(u\partial_{x^j} v) \;=\; \sum_{j=1}^n \partial_{x^j} u \partial_{x^j} v + \sum_{j=1}^n (\partial^2_{x^j x^j} u) v \;=\; \nabla u \cdot \nabla v + v\Delta u\,.$$

   Integrating both sides using the divergence theorem yields the desired identity:

$$\int_\Omega v\Delta u\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx \;=\; \int_{\partial\Omega} v\, \partial_\nu u\, dS\,.$$

2. Exchanging the roles of u and v brings

$$\int_\Omega u\Delta v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx \;=\; \int_{\partial\Omega} u\, \partial_\nu v\, dS\,.$$

   The difference between these last two expressions is Green's second identity.

Problem 2. We first write

$$\begin{aligned}\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy - u(x) \;&=\; \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy - \frac{u(x)}{|B_r(x)|}\int_{B_r(x)} dy \\ &=\; \frac{1}{|B_r(x)|}\int_{B_r(x)} \big(u(y) - u(x)\big)dy\,.\end{aligned}$$

Let $\epsilon > 0$. Since u is continuous at x by hypothesis, we find $r > 0$ such that

$$\big|u(x) - u(y)\big| \;<\; \epsilon \qquad \forall\ |x - y| < r\,.$$

In other symbols

$$\big|u(x) - u(y)\big| \;<\; \epsilon \qquad \forall\ y \in B_r(x)\,.$$

Putting this into the above integral expression yields

$$\begin{aligned}\left|\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy - u(x)\right| \;&\leq\; \frac{1}{|B_r(x)|}\int_{B_r(x)} \big|u(y) - u(x)\big|dy \\ &<\; \frac{\epsilon}{|B_r(x)|}\int_{B_r(x)} u(y)dy \;=\; \epsilon\,.\end{aligned}$$

Letting ϵ tend to 0 implies the desired identity.

Problem 3. Let $x \neq y$ be two points in I. Since f is differentiable, by the mean value theorem, there exists $z \in (x, y)$ such that

$$f'(z) \;=\; \frac{f(x) - f(y)}{x - y} \,.$$

Since f is α-H older continuous, we have

$$|f'(z)| \;\leq\; \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} |x - y|^{\alpha - 1} \;\leq\; [f]_{C^{0,\alpha}(\bar{I})} |x - y|^{\alpha-1} \,.$$

Since $\alpha > 1$, we can make the right-hand side as small as we please by choosing x and y close to each other. Hence $f'(z)$ can be made arbitrarily small and therefore f must be a constant function (vanishing derivative).

Problem 4.

1. For notational convenience, let $t = 1/p$, so we have to show

$$a^t b^{1-t} \;\leq\; \; ta + (1-t)b \,.$$

   This obviously satisfied when either $a = 0$ or $b = 0$. So without loss of generality, let $a > 0$ and $b > 0$ so that $\log(a)$ and $\log(b)$ are well-defined. Apply logarithm to both sides of the sought inequality to find that we need to show

$$t \log(a) + (1-t) \log(b) \;\leq\; \; \log\big(ta + (1-t)b\big) \,.$$

   This is the condition for the logarithm function to be **concave**. Another characterisation of concavity for twice-differentiable functions is $f'' \geq 0$. This condition is easily verified for the logarithm.

2. Put $A = a^{1/p}$ and $B = b^{1/q}$ to immediately arrive at

$$AB \;\leq\; \frac{A^p}{p} + \frac{B^q}{q} \,.$$

Problem 5.

1. This is immediate again by the divergence theorem and the harmonicity of u:

$$0 \;=\; \int_{B_\rho(y)} \Delta u \, dx \;=\; \int_{\partial B_\rho(y)} \partial_\nu u \, dS \,. \tag{1}$$

2. Perhaps we should review some basic higher dimensional spherical coordinates. Recall that in polar coordinates (r, ω) on $\mathbb{R}^2$, the area element is $dx^1 dx^2 = r dr d\omega$. In particular, on the boundary of a disk of radius ρ (i.e. a circle of radius ρ), we find $dS = \rho d\omega$. In spherical coordinates (r, θ, ϕ) on $\mathbb{R}^3$, the volume element is $dx^1 dx^2 dx^3 = r^2 \sin\theta dr d\theta d\phi$. On the boundary of a ball of radius ρ (that is on a sphere of radius ρ), we find $dS = \rho^2 \sin\theta d\theta d\phi$. We may render this as $dS = \rho^2 d\omega$ with $d\omega = \sin\theta d\theta d\phi$ representing the area element on the unit-sphere $\{|\omega| = 1\}$ in $\mathbb{R}^3$. This pattern continues in higher dimension. For example, on $\mathbb{R}^n$, the integration element on ∂B_ρ takes the form $dS = \rho^{n-1} d\omega$, where $d\omega$ represents the integration element on $\{|\omega| = 1\}$ in $\mathbb{R}^n$. Accordingly, we can recast (3) as

$$0 \;=\; \rho^{n-1} \int_{|\omega|=1} (\partial_r u)(y + \rho\omega) \, d\omega \,. \tag{2}$$

We have used the fact that on the boundary $|\omega| = 1$, the outward unit normal points in the radial direction (straight out of the ball), Since r and ω are independent variables, we are free to move ∂_r outside of the integral sign. This implies now

$$0 \;=\; \rho^{n-1}\partial_\rho \int_{|\omega|=1} u(y+\rho\omega)\,d\omega\;.$$

Switching back to the area element dS yields finally

$$0 \;=\; \rho^{n-1}\partial_\rho\left(\rho^{1-n}\int_{\partial B_\rho(y)} u(x)\,dS\right)\;.$$

3. The previous question shows that the following quantity has zero derivative and must be constant:

$$\rho^{1-n}\int_{\partial B_\rho(y)} u(x)\,dS \qquad \forall\;\rho\in(0,R)\;.$$

Letting ρ tend to 0 and to R, we reach the identity

$$\lim_{\rho\to 0}\,\rho^{1-n}\int_{\partial B_\rho(y)} u(x)\,dS \;=\; R^{1-n}\int_{\partial B_R(y)} u(x)\,dS\;.$$

Next we use the given hint[1] to recast this as

$$\lim_{\rho\to 0}\frac{1}{|\partial B_\rho(y)|}\int_{\partial B_\rho(y)} u(x)\,dS \;=\; \frac{1}{|\partial B_R(y)|}\int_{\partial B_R(y)} u(x)\,dS\;.$$

Finally, calling upon Problem 2, we arrive at the mean value property for harmonic functions:

$$u(y) \;=\; \frac{1}{|\partial B_R(y)|}\int_{\partial B_R(y)} u(x)\,dS\;.$$

4. In the case of a subharmonic function, equation (3) becomes an inequality:

$$0 \;\le\; \int_{\partial B_\rho(y)} \partial_\nu u\,dS\;. \tag{3}$$

Following *mutatis mutandis* the same line of reasoning as above, we arrive eventually at

$$0 \;\le\; \partial_\rho\left(\rho^{1-n}\int_{\partial B_\rho(y)} u(x)\,dS\right)\;.$$

Hence

$$\rho^{1-n}\int_{\partial B_\rho(y)} u(x)\,dS$$

is now a non-decreasing function of ρ, and in particular

$$\lim_{\rho\to 0}\,\rho^{1-n}\int_{\partial B_\rho(y)} u(x)\,dS \;\le\; R^{1-n}\int_{\partial B_R(y)} u(x)\,dS\;.$$

Proceeding as we did above, we arrive at the mean value property for subharmonic functions:

$$u(y) \;\le\; \frac{1}{|\partial B_R(y)|}\int_{\partial B_R(y)} u(x)\,dS\;.$$

[1] Prove the hint is true!