Self-assess quiz for AMSI Summer School 2026 course Mathematical Optimisation Theory and Application

Dr. Hoa T. Bui
Curtin Centre for Optimisation and Decision Science

Dr. Vinesha Peiris
Curtin Centre for Optimisation and Decision Science

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As necessary background for the course, you will need to have taken undergraduate courses in real analysis, basic vector calculus and linear algebra. It is also desirable to have some familiarity with a computer programming language like Matlab, Julia, Python, etc. In the course, we will be using Python. The following questions will help you assess your readiness for the course. Note: some of these questions and answers were prepared with the help of Dr. Scott Lindstrom.

1 Quiz

1. Let $p : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Use the Cauchy–Schwarz inequality to show that p is a norm in $\mathbb{R}^n$. It is completely fine if you need to look up the definition of the Cauchy–Schwarz inequality.

2. Let $p : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \sqrt[q]{x_1^q + x_2^q + \cdots + x_n^q}$. Show that if $q \in (0,1)$, then p is not a norm in $\mathbb{R}^n$.

3. Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ be defined as

$$d(x,y) := \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

   Prove that (X,d) is a metric space.If X is a vector space and $X \setminus \{0\} \neq \emptyset$, (X,d) is not associated with any norm.

4. Compute the gradient $\nabla f(x,y)$ where $f(x,y) = x^2y + ye^{xy}$.

5. For the given data points $(0,1), (1,2), (2,5), (3,10), (4,17)$, find the best-fit quadratic function $y(t) = at^2 + bt + c$ using least squares.

6. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be invertible. Can the inverse $\mathbf{A}^{-1}$ be unique?

7. Consider the matrices

$$A := \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \quad \text{and} \quad B := \begin{vmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{vmatrix}.$$

   (a) Show that A is invertible, or explain why it is not invertible.
   (b) Show that B is invertible, or explain why it is not invertible.
   (c) Calculate $A + B$ or explain why it does not exist.
   (d) Calculate AB or explain why AB does not exist.
   (e) Calculate BA or explain why BA does not exist.
   (f) Calculate $B^T A$ or explain why $B^T A$ does not exist.

2 Solutions

1. We show p is a norm.

   - **Positive definite:** Notice that $p(x) = 0$ if and only if $x_1 = x_2 = \cdots = x_n = 0$, in which case $x = 0$.
   - **Absolute homogeneity**: Let $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} p(\lambda x) &= \sqrt{(\lambda x_1)^2 + \cdots + (\lambda x_n)^2} \\ &= \sqrt{\lambda^2(x_1^2 + \cdots + x_n^2)} = |\lambda|\sqrt{x_1^2 + \cdots + x_n^2} = |\lambda| p(x). \end{aligned}$$

   - **Triangle inquality**: Let $x, y \in \mathbb{R}^n$. From the Cauchy-Schwarz inequality, it holds that

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \le \left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right),$$

   or equivalently

$$\sum_{i=1}^n x_i y_i \le \sqrt{\sum_{i=1}^n x_i^2} \cdot \sqrt{\sum_{i=1}^n y_i^2} = p(x) \cdot p(y).$$

   Then

$$\begin{aligned} p(x+y)^2 &= \sum_{i=1}^n (x_i + y_i)^2 \\ &= \sum_{i=1}^n x_i^2 + y_i^2 + 2x_i y_i \\ &= p(x)^2 + p(y)^2 + 2\sum_{i=1}^n x_i y_i \\ \text{(Cauchy-Schwarz)} \quad &\le p(x)^2 + p(y)^2 + 2p(x)p(y) \\ &= (p(x) + p(y))^2. \end{aligned}$$

2. Take $q \in (0,1)$. Consider $x = (1,0,0,\ldots,0) \in \mathbb{R}^n$ and $y = (0,1,0,\ldots,0) \in \mathbb{R}^n$. We have

$$p(x) = p(y) = 1, \quad p(x+y) = (1^q + 1^q)^{1/q} = (2)^{1/q}.$$

Since $q \in (0,1)$, $1/q > 1$ and so $2^{1/q} > 2$. This implies

$$p(x+y) > 2 = p(x) + p(y),$$

which disproves the triangle inequality. Hence, p is not a norm.

3. Proof for (X,d) is a metric space.

   (a) *Identity of indiscernible.* For all $x, y \in X$, $\quad d(x,y) = 0 \Longleftrightarrow x = y$.

   (b) *Symmetry.* For all $x, y \in X$,

   $$d(x,y) = d(y,x) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

   (c) *Triangle inequality.* For all $x, y, z \in X$, we consider two cases
   Case 1. $x = y = z$. Then, due to *identity of indiscernible* property, we have

   $$d(x,y) = 0 = d(x,z) + d(z,y).$$

   Case 2. $x \neq y$ or $x \neq z$. Then, due to the *identity of indiscernible* property, and $1 \geq d(a,b) \geq 0$ for all $a, b \in X$,

   $$d(x,z) + d(z,y) \geq 1 \geq d(x,y).$$

   Therefore, (X,d) is a metric.

   Suppose X is a vector space. We prove that (X,d) is not a norm space by contradiction. Suppose there is a norm $\|\cdot\| : X \to \mathbb{R}$ that associates with metric d, i.e., for all $x, y \in X$,

   $$\|x - y\| = d(x,y).$$

   Take $x \in X \setminus \{0\}$. By the *homogenity* properties, and the definition of d

   $$\|2x\| = 2\|x\| = 2\|x - 0\| = 2d(x,0) = 2.$$

   On the other hand,

   $$\|2x\| = \|3x - x\| = d(3x,x) \leq 1,$$

   which is a contradiction.

4. The gradient consists of the partial derivatives in the variables x and y as follows:

   $$\nabla f(x,y) = \left(2xy + y^2 e^{xy}, x^2 + yxe^{xy} + e^{xy}\right)$$

5. Construct the design matrix $\mathbf{X}$ using each t_i, where each row of $\mathbf{X}$ is $[t_i^2, t_i, 1]$. So, the design matrix X and the output vector y are:

$$X = \begin{bmatrix} 0^2 & 0 & 1 \\ 1^2 & 1 & 1 \\ 2^2 & 2 & 1 \\ 3^2 & 3 & 1 \\ 4^2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 10 \\ 17 \end{bmatrix}$$

Then solve the normal equations:

$$X^\top X \begin{bmatrix} a \\ b \\ c \end{bmatrix} = X^\top y$$

$$X^\top X = \begin{bmatrix} 354 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 5 \end{bmatrix}, \quad X^\top y = \begin{bmatrix} 384 \\ 110 \\ 35 \end{bmatrix}$$

So the linear system to solve is:

$$\begin{bmatrix} 354 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 384 \\ 110 \\ 35 \end{bmatrix}$$

Solving this system (e.g., by Gaussian elimination or a numerical method), we find:

$$a = 1, \quad b = 1, \quad c = 1$$

So the best-fit quadratic is:

$$y(t) = t^2 + t + 1$$

6. Suppose there exist two inverses B and C such that:

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

and

$$\mathbf{AC} = \mathbf{CA} = \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix. Then:

$$\mathbf{B} = \mathbf{B}\,\mathbf{I} = \mathbf{B}(\mathbf{AC}) = (\mathbf{BA})\mathbf{C} = \mathbf{IC} = \mathbf{C}.$$

Hence, $\mathbf{B} = \mathbf{C}$. So the inverse is unique.

7. - A is invertible, because its column vectors are linearly independent. To verify this, suppose that $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ where v_1, v_2, v_3 are the column vectors of A from left to right and $x_1, x_2, x_3 \in \mathbb{R}$. This is just $Ax = 0$, and so we obtain the system

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 10 & 0 \end{array}\right|$$

We can put this in reduced row echelon form as follows:

$$\left|\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right|,$$

which shows that $x = 0$.

- B cannot be invertible, because it is not a square matrix.
- $A + B$ does not exist because A and B do not have the same dimensions.
- We calculate AB by matrix multiplication:

$$AB = \left|\begin{array}{cc} 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 & 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 \\ 4 \cdot 1 + 5 \cdot 3 + 6 \cdot 5 & 4 \cdot 2 + 5 \cdot 4 + 6 \cdot 6 \\ 7 \cdot 1 + 8 \cdot 3 + 10 \cdot 5 & 7 \cdot 2 + 8 \cdot 4 + 10 \cdot 6 \end{array}\right|.$$

- BA does not exist because the number of rows of B does not match the number of columns in A.
- $B^T A$ is computed as follows:

$$B^T A = \left|\begin{array}{ccc} 1 \cdot 1 + 3 \cdot 2 + 5 \cdot 3 & 1 \cdot 4 + 3 \cdot 5 + 5 \cdot 6 & 1 \cdot 7 + 3 \cdot 8 + 5 \cdot 10 \\ 2 \cdot 1 + 4 \cdot 2 + 6 \cdot 3 & 2 \cdot 4 + 4 \cdot 5 + 6 \cdot 6 & 2 \cdot 7 + 4 \cdot 8 + 6 \cdot 10 \end{array}\right|.$$

3 Pre-reading

1. Convex Optimization by Stephen Boyd and Lieven Vandenberghe, Cambridge University Press;
2. Deep learning by Yoshua Bengio, Ian Goodfellow, Aaron Courville;
3. Wolsey, Laurence A. Integer programming. John Wiley & Sons, 2020.