The following questions assume you have taken a first course in linear algebra and in abstract algebra, including group theory, and have some familiarity with rings and fields.

In the exercises below, $\mathbb{Z}_n$ will denote the ring of integers modulo n.

1. Assume V, W are finite dimensional vector spaces over the field $\mathbb{F}$, and let $f\colon V \to W$ be a linear map. Define the *cokernel* of f as the quotient $coker(f) = W/Im(f)$.

(i) Prove f is surjective if and only if $coker(f) = 0$.

(ii) Prove that $\dim(coker(f)) + \dim(rk(f)) = \dim(W)$.

(iii) Determine a basis over $\mathbb{Z}_2$ for the cokernel of the map $f\colon (\mathbb{Z}_2)^3 \to (\mathbb{Z}_2)^3$ represented by the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

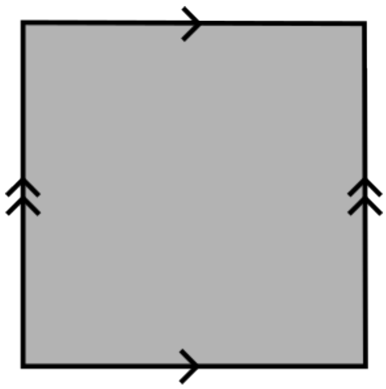
2. *(i)* Let $m, n \in \mathbb{Z}$ be coprime. Show there is no nontrivial homomorphism between $\mathbb{Z}_m$ and $\mathbb{Z}_n$.

(ii) Prove that $\mathbb{Z}_n$ is a field if and only if n is prime.

(iii) Prove that, if p is prime, $(\mathbb{Z}_p)^n$ is a vector space.

(iv) Prove that the cokernel of the map $\cdot p\colon \mathbb{Z} \to \mathbb{Z}$, defined as multiplication by a fixed prime $p \in \mathbb{Z}$ is isomorphic to $\mathbb{Z}_p$.

3. Consider the square in the figure below, with the left/right and top/bottom identifications shown; call the resulting shape T.



(i) Draw T.

(ii) Consider the group quotient $\mathbb{R}^2/\mathbb{Z}^2$, corresponding to the equivalence relation $(x, y) \sim (x', y')$ if and only if $(x - x', y - y') \in \mathbb{Z}^2$. Argue that there exists a "natural" bijection between points in T and equivalence classes in the quotient $\mathbb{R}^2/\mathbb{Z}^2$.

(iii) Draw the image in T of the line $2y = 3x$ in $\mathbb{R}^2$ under the bijection from point *(ii)*.

4. Let $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\} \subset \mathbb{C}$ be the unit circle. Prove that S^1 is a multiplicative subgroup of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and describe the cosets of S^1. Prove that $\mathbb{C}^*/S^1 \cong \mathbb{R}$.