Applied Partial Differential Equations in Fluid Dynamics

Pre-enrolment Quiz

Vector calculus

1. Calculate the divergence of the following vector fields:

   (a) $\boldsymbol{u} = (2x, 0)$

   (b) $\boldsymbol{u} = (x^2 + y^2, -2xy)$

   (c) $\boldsymbol{u} = (f(y), 0)$, for any differentiable function $f(y)$.

2. Consider a curve represented by a function $y = f(x)$ in the plane.

   (a) Find expressions for the unit (right-pointing) tangent $\boldsymbol{t}$ and (upward-pointing) normal vector $\boldsymbol{n}$ at each point along the curve, in terms of $f'(x)$.

   (b) The normal and tangent are related through the Frenet formula:

   $$\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = k\boldsymbol{n}$$

   where s is the arclength and k is the curvature of the curve. Show that this holds given the curvature is defined as

   $$k = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}.$$

ODEs and Stability

3. Consider the nonlinear system of differential equations

   $$\frac{\mathrm{d}u}{\mathrm{d}t} = v - u^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \alpha - u,$$

   where $\alpha > 0$ is a parameter.

   (a) Find the unique fixed point (i.e. steady state, equilibrium) of the system.

   (b) Explain (in words) why the eigenvalues of the Jacobian of the system evaluated at the fixed point determine the stability of the fixed point.

(c) By finding these eigenvalues, determine the stability of this fixed point (noting any dependence on the parameter α).

4. The following boundary value problem describes fluid velocity $u(y)$ between two solid plates, driven by a pressure difference:

$$\mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = \Delta p, \qquad 0 < y < H.$$

Here Δp is the pressure difference per unit length, μ is the viscosity of the liquid, and H is the plate separation. The velocity is zero on each wall:

$$u(0) = u(H) = 0.$$

(a) Determine the velocity profile $u(y)$.

(b) The flux q is defined to be

$$q = \int_0^H u(y)\,\mathrm{d}y.$$

Calculate the flux as a function of the pressure difference, the viscosity, and the plate separation.

Numerical ODE Calculation

5. Consider the ODE system from Question 3:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = v - u^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \alpha - u.$$

By choosing your own value of α and initial condition, use one of the many numerical ODE solvers in your favourite programming language, e.g. MATLAB, Julia, Python (SciPy), to compute the solution of this ODE. See if the solution agrees both qualitatively with your analysis in Question 3, and also quantitatively (e.g, can you observe your predicted growth/decay near the steady state, at the predicted rate?)

Partial Differential Equations

6. Consider the PDE (the heat equation):

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L,$$

with boundary conditions $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ and initial condition $u(x,0) = f(x)$.

(a) Show that if we define new variables

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{D}{L^2}t, \qquad \hat{u}(\hat{x},\hat{t}) = \frac{u(x,t)}{M},$$

where $M = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \neq 0$, then the problem for $\hat{u}$ is

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}, \qquad 0 < \hat{x} < 1, \qquad \frac{\partial \hat{u}}{\partial \hat{x}}(0, \hat{t}) = \frac{\partial \hat{u}}{\partial \hat{x}}(1, \hat{t}) = 0 \tag{1}$$

and initial condition

$$\hat{u}(\hat{x}, 0) = \hat{f}(\hat{x}), \qquad \int_0^1 \hat{f}(\hat{x})\,\mathrm{d}\hat{x} = 1.$$

(b) Assume the PDE above (1) has a solution of the form

$$\hat{u}(\hat{x}, \hat{t}) = \mathrm{e}^{\lambda \hat{t}} \cos(k\hat{x}).$$

Determine what k must be to satisfy the boundary conditions, and what the corresponding λ must be to satisfy the PDE (neglecting the initial condition).

(c) Using the above result, and/or your other knowledge (e.g. Fourier series, separation of variables) find the series solution to the PDE for a given initial condition $\hat{f}(\hat{x})$.

(d) What is the long-time behaviour of the solution (for any initial condition)?

Solutions

1. (a) $\nabla \cdot \boldsymbol{u} = 2$
   (b) $\nabla \cdot \boldsymbol{u} = 2x - 2x = 0$.
   (c) $\nabla \cdot \boldsymbol{u} = \frac{\partial}{\partial x} f(y) = 0$.
2. (a) Tangent vector is tangent to curve, so $\boldsymbol{t} \propto (1, f'(x))$. Normalising:
$$\boldsymbol{t} = \frac{(1, f'(x))}{\sqrt{1 + f'(x)^2}}.$$
   Normal is orthogonal to tangent vector, so $\boldsymbol{n} \propto (-f'(x), 1)$. Normalising:
$$\boldsymbol{n} = \frac{(-f'(x), 1)}{\sqrt{1 + f'(x)^2}}.$$
   (b) Arclength s defined by
$$\frac{\mathrm{d}s}{\mathrm{d}x} = \sqrt{1 + f'(x)^2}$$
   So
$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} &= \frac{1}{\sqrt{1 + f'(x)^2}} \frac{\mathrm{d}}{\mathrm{d}x} \left[\frac{(1, f'(x))}{\sqrt{1 + f'(x)^2}} \right] \\
&= \frac{1}{\sqrt{1 + f'(x)^2}} \left[\frac{-f'(x) f''(x)}{(1 + f'(x)^2)^{3/2}}, \frac{f''(x)(1 + f'(x)^2) - f'(x)^2 f''(x)}{(1 + f'(x)^2)^{3/2}} \right] \\
&= \frac{f''(x)}{(1 + f'(x)^2)^{3/2}} \left[\frac{(-f'(x), 1)}{\sqrt{1 + f'(x)^2}} \right] = k\boldsymbol{n}.
\end{aligned}$$
3. (a) Fixed points when $u' = v' = 0$, or $u = \alpha$, $v = \alpha^2$.
   (b) Let $[u, v]^T = \boldsymbol{u}$ and ODE system is $\boldsymbol{u}' = \boldsymbol{F}(\boldsymbol{u})$. Linearisation of system near fixed point $\boldsymbol{u}^*$ results in
$$\boldsymbol{F}(\boldsymbol{u}) \approx \boldsymbol{F}(u^*) + J(\boldsymbol{u}^*)(\boldsymbol{u} - \boldsymbol{u}^*)$$
   Define $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}^*$ be difference between solution and fixed point. Then
$$\frac{\mathrm{d}\tilde{\boldsymbol{u}}}{\mathrm{d}t} \approx J(\boldsymbol{u}^*)\tilde{\boldsymbol{u}}.$$
   The general solution of this linear, constant coefficient system is given in terms of eigenvalues λ and eigenvectors $\boldsymbol{v}$ of J as
$$\tilde{\boldsymbol{u}} = c_1 \boldsymbol{v}_1 \mathrm{e}^{\lambda_1 t} + c_2 \boldsymbol{v}_2 \mathrm{e}^{\lambda_2 t},$$
   (ignoring special cases such as repeated eigenvalues). If both $\Re(\lambda) < 0$, $\tilde{\boldsymbol{u}} \to \boldsymbol{0}$ for any initial condition (so fixed point is asymptotically stable). Technically this is a stable node if λ are real, stable spiral if λ are complex conjugate.

If at least one $\Re(\lambda) > 0$ fixed point is unstable (technically saddle when one is positive and one is negative, unstable node if both positive and real, unstable spiral when complex conjugate).

(Linearisation insufficient if any $\Re(\lambda) = 0$; *Hartman–Grobman theorem* makes this rigorous).

(c) Jacobian is

$$J = \begin{bmatrix} -2u & 1 \\ -1 & 0 \end{bmatrix} \quad \Rightarrow \quad J(\boldsymbol{u}^*) = \begin{bmatrix} -2\alpha & 1 \\ -1 & 0 \end{bmatrix}.$$

Eigenvalues of this matrix are solutions of $\lambda^2 + 2\alpha\lambda + 1 = 0$, thus

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - 1}.$$

The fixed point is thus stable for all $\alpha > 0$; for $\alpha < 1$, a spiral, and for $\alpha \geq 1$, a node.

4. (a) Integrate $u'' = \dfrac{\Delta p}{\mu}$ twice to find

$$u = \frac{\Delta p}{2\mu} y^2 + c_1 y + c_2.$$

Applying boundary conditions, $c_2 = 0$ and $\dfrac{\Delta p}{2\mu} H^2 + c_1 H = 0$, so

$$c_1 = -\frac{\Delta p}{2\mu} H,$$

and

$$u(y) = -\frac{\Delta p}{2\mu} y(H - y)$$

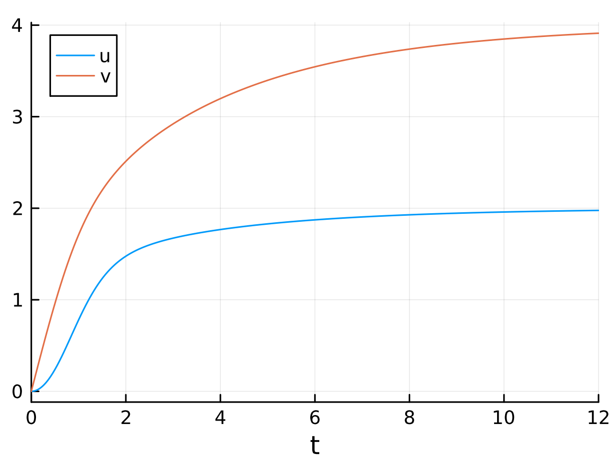
(b) Integrate the velocity over $[0, H]$ to find the flux:

$$q = -\frac{\Delta p}{2\mu} \int_0^H Hy - y^2 \, \mathrm{d}y = -\frac{\Delta p}{2\mu} \left[\frac{Hy^2}{2} - \frac{y^3}{3} \right]_0^H = -\frac{\Delta p H^3}{12\mu}.$$

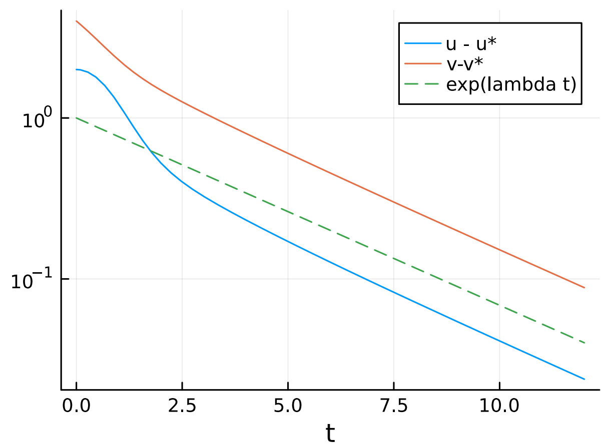
5. Example code from Julia:

```
using DifferentialEquations, Plots

function ODEFun!(du, u, p, t)
    alpha = p[1]
    @. du = [u[2] - u[1]^2, alpha - u[1]]
end

u0 = [0., 0.]
tspan = (0., 12.)
alpha = 2.

p = [alpha]
prob = ODEProblem(ODEFun!, u0, tspan, p)
sol = solve(prob, reltol=1e-8)

#subtract steady state
udiff = [abs(u[1] - alpha) for u in sol.u]
vdiff = [abs(u[2] - alpha^2) for u in sol.u]

#larger of the (real part of) eigenvalues
if abs(alpha) > 1
    lambda = -alpha + sqrt(alpha^2 - 1)
else
    lambda = -alpha
end

plt1 = plot(sol, labels=["u" "v"], size=(400, 300))
savefig(plt1, "preEnrolmentQuizSols.pdf")

plt2 = plot(sol.t, udiff, yaxis=:log, label="u - u*", xlabel="t", size=(400, 300
    ))
plot!(plt2, sol.t, vdiff, label="v-v*")
plot!(plt2, sol.t, exp.(sol.t*lambda), linestyle=:dash, label="exp(lambda t)")
#plot!(labels=["u" "v" "exp(lambda t)"])

savefig(plt2, "preEnrolmentQuizDecay.pdf")
```

Plots generated from code (for $\alpha = 2$) show the solution tends to $(u^*, v^*) = (\alpha, \alpha^2)$ monotonically (since $\alpha > 1$, eigenvalues real):



We can observe the solution decays to the fixed point exponentially by plotting the difference on a logarithmic axis, and comparing to the exponential with coefficient equal to the larger eigenvalue $\lambda = -\alpha + \sqrt{\alpha^2 - 1}$.



6. (a) Substitute in new variables (using chain rule for derivatives):

$$\frac{MD}{L^2}\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{MD}{L^2}\frac{\partial^2 u}{\partial x^2} \qquad \Rightarrow \qquad \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}.$$

Domain translated to $0 < \hat{x} < 1$. Homogeneous boundary conditions are unchanged. Initial condition is

$$\hat{u}(x, 0) = \frac{f(x)}{M} = \hat{f}(\hat{x}).$$

From definition of M, we have

$$\int_0^1 \hat{f}(\hat{x})\,\mathrm{d}\hat{x} = \frac{1}{M}\int_0^1 f(x)\,\frac{\mathrm{d}x}{L} = \frac{M}{M} = 1.$$

(b) Assume $\hat{u} = \mathrm{e}^{\lambda \hat{t}} \cos(k\hat{x})$. Boundary conditions require

$$\sin(0) = \sin(k) = 0 \qquad \Rightarrow \qquad k = n\pi,\ n = 0, 1, 2, \ldots.$$

From the PDE we have $\lambda = -k^2$.

(c) Since the PDE and BCs are homogeneous we can (formally) take the linear combination over all solutions, so that the general solution is

$$\hat{u}(\hat{x}, \hat{t}) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{-n^2\pi^2\hat{t}} \cos(n\pi\hat{x}).$$

The coefficients a_n are given by the Fourier cosine coefficients of the initial condition, that is

$$a_0 = \int_0^1 \hat{f}(\hat{x})\,\mathrm{d}\hat{x} = 1, \qquad a_n = 2\int_0^1 \hat{f}(\hat{x})\cos(n\pi\hat{x})\,\mathrm{d}\hat{x}$$

(note $a_0 = 1$ follows from the way PDE was nondimensionalised).

(d) For any initial condition, terms with negative exponentials decay to zero as $\hat{t} \to \infty$, so the only remaining term is the constant (with $a_0 = 1$):

$$\hat{u}(\hat{x}, \hat{t}) \to 1, \qquad \hat{t} \to \infty.$$