THE UNIVERSTY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Pre-enrolment Quiz
Introduction to Financial Calculus

AMSI Summer School 2025
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1. Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with the same distribution.

(a) Is it true that

$$\mathbb{E}\left(\frac{X}{X+Y}\right) = \mathbb{E}\left(\frac{Y}{X+Y}\right)?$$

(b) Suppose additionally that $X \leq Y$. Is that true that $X = Y$ a.s.?

If the answer is "yes" prove the claim, if the answer is "no" provide a counterexample.

2. Suppose we independently roll two standard 6-sided dice. Let X_1 and X_2 the observed number in the first and second dice respectively. Denote by $\sigma(Y)$ the σ-algebra generated by a random variable Y.

(a) Compute expectations $\mathbb{E}[X_1 + X_2]$ and $\mathbb{E}[X_1]$.

(b) Compute conditional expectations $\mathbb{E}[X_1 + X_2|\sigma(X_1)]$ and $\mathbb{E}[X_1|\sigma(X_1 + X_2)]$.

3. Let X be a random variable with normal distribution $\mathcal{N}(0, 1)$.

(a) Show that $a + bX$, where $a, b \in \mathbb{R}$ has the normal distribution $\mathcal{N}(a, b^2)$.

(b) Find the density of $Y = e^X$. (The distribution of Y is called the log normal distribution.)

4. A random variable X has exponential distribution with parameter $\alpha > 0$ if its distribution function is

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Let X_1 and X_2 be two independent random variables with exponential distribution with parameters $\alpha > 0$ and $\beta > 0$ respectively. Find the distribution function and probability density function of $\min\{X_1, X_2\}$. Compute the probability $\mathbb{P}(X_1 < X_2)$.

(b) Random variables X and Y are independent and have exponential distribution with the same parameter α. Find the distribution of $X - Y$.

5. Let $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$ and $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. Let $\mathcal{Y}$ be a sub σ algebra and X be a random variable given by:

$$\mathcal{Y} = \sigma([0, \frac{1}{4}], [\frac{1}{4}, 1]), \quad \text{and} \quad X(\omega) = \sqrt{\omega}.$$

Compute the conditional expectation $\mathbb{E}(X|\mathcal{Y})$.

6. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}$ be a filtration on that space. Suppose that τ_1 and τ_2 are two stopping times.

(a) Show that $\tau_1 + \tau_2$ is a stopping time.

(b) Assume that $\tau_2 \geq \tau_1$. Is it true that $\tau_2 - \tau_1$ is a stopping time? Either prove that it is or provide an example showing that it is not.