PRE-ENROLMENT QUIZ FOR THE 2025 AMSI SUMMER COURSE "OPTIMAL TRANSPORTATION AND MONGE-AMPÈRE EQUATIONS

Problem 1. (Analysis) Suppose u is a smooth, convex function defined on $B_4(0)$. Let $v = u + \frac{1}{2}|x|^2$. Show that the volume $|Dv(B_1(0))| \geq |B_1(0)|$.

Problem 2. (Analysis) Let $u \in C^1$ be a convex function satisfying that

$$\left| u(x) - \frac{1}{2}|x|^2 \right| < \delta, \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain containing a unit ball inside, namely $B_1 \Subset \Omega$, and $\delta \in (0, \frac{1}{100})$ is a small constant. Prove that

$$|Du(x) - x| < 2\sqrt{\delta},$$

for all $x \in \Omega$ satisfying $\mathrm{dist}(x, \partial\Omega) > \frac{1}{5}$.

Problem 3. (Geometry) Consider the surface {(x,y,F(x,y))}, where $F : \mathbb{R}^2 \to \mathbb{R}$ is smooth and convex. How to evaluate the Gaussian curvature at the general point $(x, y, F(x, y))$?

Problem 4. (Algebra) Let M be a real, invertible $n \times n$ matrix. Prove that M can be decomposed to a product $M = PO$ such that O is an $n \times n$ orthogonal matrix and P is an $n \times n$ positive-definite, symmetric matrix.

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