Algebraic Knot Theory Pre-Quiz

The following questions assume you have completed the pre-reading, or have taken a first course in abstract algebra (including group theory and some familiarity with rings). The quiz should take up to 90 minutes (more if you need to look up concepts as you go, then don't worry about time).

1. Let $\mathbb{Z}_n$ denote the additive group of modulo n integers. (You may have seen this group denoted $\mathbb{Z}/n$ or $\mathbb{Z}/n\mathbb{Z}$.)

(a) How many group homomorphisms are there from $\mathbb{Z}_{18}$ to $\mathbb{Z}_{24}$?

(b) What is the maximal size of the image of a group homomorphism $\varphi : \mathbb{Z}_{18} \to \mathbb{Z}_{24}$? How many different homomorphisms have image of this maximal size?

2. Let G, H be groups, let 1_H denote the identity element of H. Let $f : G \to H$ be a group homomorphism, and $N = \{g \in G \mid f(g) = 1_H\}$.

(a) Prove that N is a normal subgroup of G.

(b) What is the subgroup N usually called?

3. (a) Give a *generators and relations*[1] presentation of the dihedral group D_6 (the group of symmetries of a hexagon).

(b) Use your answer to part (a) to find the number of homomorphisms from D_6 to $\mathbb{Z}_{12}$.

4. Let R be a ring and I an ideal[2] in R. Let $I^k = \{ri_1 i_2 \cdots i_k \mid r \in R, i_s \in I \text{ for } s = 1, 2, ..., k\}$. Prove that I^k is also an ideal in R, and $I^k \supseteq I^{k+1}$.

5. An *algebra* is a ring and a vector space at the same time. That is, an algebra over a field $\mathbb{F}$ is a set A along with

- two binary operations, addition $+$ and multiplication $*$, on A, so that $(A, +, *)$ is a ring;
- and a scalar multiplication ($\cdot : \mathbb{F} \times A \to A$), so that $(A, +, \cdot)$ is a vector space over $\mathbb{F}$.

(a) Can you think of two good examples of algebras over the complex numbers?
(What can you add together, multiply together, and also multiply by complex numbers?)

(b) Let G be a group, and define

$$\mathbb{C}G = \left\{ \sum_{i=1}^{n} \alpha_i g_i \,\middle|\, n \in \mathbb{Z}; \alpha_i \in \mathbb{C}, g_i \in G \text{ for } i = 1, \ldots, n \right\}.$$

The set $\mathbb{C}G$ comes with a natural addition and scalar multiplication over $\mathbb{C}$. Extend the group multiplication of G linearly[3] to define a multiplication on $\mathbb{C}G$. Show that $\mathbb{C}G$ is an algebra over $\mathbb{C}$. (This is called the *group algebra* of G.)

(c) *Challenge (optional):* Identify the group algebra $\mathbb{C}\mathbb{Z}$ as a familiar algebra (a set of mathematical objects with naturally defined addition, multiplication and scalar multiplication operations). (*Hint: it's easier if you think of $\mathbb{Z}$ as a multiplicative group.*)

[1]If you're not familiar with generators and relations presentations, see Chapters 1.2 and 6.3 of Dummit and Foote, or Chapters 7.9-7.10 of Artin.

[2]If unfamiliar, see Dummit and Foote Chapter 7.1–3, or Artin Chapter 11.1–3.

[3]For example, if $g, h \in G$, then $(2g - 3h)(5h + g) = 2g^2 + 10gh - 3hg - 15h^2 \in \mathbb{C}G$. Note that unless G is abelian, the gh and hg terms do not necessarily simplify, since possibly $gh \neq hg$.

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