# Warm-up questions

1. The divergence of a (smooth) vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathrm{div} V(x) := \sum_{j=1}^{n} \frac{\partial V_j}{\partial x_j}(x),$$

with the notation $V(x) = (V_1(x), \dots, V_n(x))$.

The gradient of a (smooth) function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\nabla f(x) := \left( \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right).$$

The Laplacian of a (smooth) function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\Delta f(x) := \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}(x).$$

Prove that, given a (smooth) vector field $V : \mathbb{R}^n \to \mathbb{R}^n$ and (smooth) functions $f$, $g : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{div}(\nabla f) = \Delta f, \tag{1}$$

$$\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g, \tag{2}$$

$$\Delta(\mathrm{div} V) = \mathrm{div}(\nabla(\mathrm{div} V)). \tag{3}$$

2. If $V_n$ denotes the volume of the ball of radius 1 in $\mathbb{R}^n$ and $A_n$ denotes the surface area of the unit sphere (i.e., the boundary of the ball of radius 1 in $\mathbb{R}^n$), prove that

$$V_n = \frac{A_n}{n}.$$

3. Assume that $f$ is continuous in $(-1, 1)$ and differentiable in $(-1, 0) \cup (0, 1)$.

Suppose that

$$\lim_{x \searrow 0} f'(x) = \lim_{x \nearrow 0} f'(x) = 0.$$

Prove that $f$ is differentiable at 0 and calculate $f'(0)$.

4. Calculate

$$\lim_{x\to 0}\left(\cos x + e^x \sin x + e^{x^2} - e^{x^4}\right)^{\frac{1}{\ln(1+x)}}.$$

5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous if there exists $M > 0$ such that, for every $x$, $y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq M|x - y|. \tag{4}$$

Given two Lipschitz continuous functions $f$, $g : \mathbb{R} \to \mathbb{R}$, let

$$h(x) := \max\{f(x), g(x)\}.$$

Prove that $h$ is Lipschitz continuous.

# Solutions

1. We have that

$$\mathrm{div}(\nabla f) = \mathrm{div}(\partial_{x_1} f, \dots, \partial_{x_n} f) = \sum_{j=1}^{n} \partial_{x_j}(\partial_{x_j} f) = \sum_{j=1}^{n} \partial^2_{x_j} f = \Delta f,$$

which is (1).

Moreover, by (1),

$$\begin{aligned}\Delta(fg) &= \mathrm{div}(\nabla(fg))\\ &= \mathrm{div}(\partial_{x_1}(fg), \dots, \partial_{x_n}(fg))\\ &= \mathrm{div}(f\partial_{x_1} g + g\partial_{x_1} f, \dots, f\partial_{x_n} g + g\partial_{x_n} f)\\ &= \sum_{j=1}^{n} \partial_{x_j}\big(f\partial_{x_j} g + g\partial_{x_j} f\big)\\ &= \sum_{j=1}^{n} \big(f\partial^2_{x_j} g + \partial_{x_j} f\partial_{x_j} g + g\partial^2_{x_j} f + \partial_{x_j} f\partial_{x_j} g\big)\\ &= f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g,\end{aligned}$$

proving (2).

Also, by applying (1) to $f := \mathrm{div}V$,

$$\mathrm{div}(\nabla(\mathrm{div}V)) = \mathrm{div}(\nabla f) = \Delta f = \Delta(\mathrm{div}V),$$

which is (3).

2. Simple examples confirming the desired result are $n = 2$ (since $V_2 = \pi$ and $A_2 = 2\pi$) and $n = 3$ (since $V_3 = \frac{4\pi}{3}$ and $A_3 = 4\pi$); even the case $n = 1$ would work if understood properly (the boundary of the interval consisting of two points, giving $V_1 = 2$ and $A_1 = 2$).

   The general case can be obtained, for example, by polar coordinates in $\mathbb{R}^n$: indeed, if $B_1$ is the ball of unit radius, we have

$$V_n = \int_{B_1} dx = A_n \int_0^1 \varrho^{n-1}\, d\varrho = \frac{A_n}{n}.$$

3. We know that for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that if $x \in [-\delta, 0) \cup (0, \delta]$ then $|f'(x)| \leq \epsilon$.

   Let now $h \in (0, \delta)$. Then, by the Fundamental Theorem of Calculus,

$$|f(h) - f(0)| = \left| \int_0^h f'(t)\, dt \right| \leq \int_0^h |f'(t)|\, dt \leq \epsilon h.$$

   Similarly, if $h \in (-\delta, 0)$,

$$|f(0) - f(h)| = \left| \int_h^0 f'(t)\, dt \right| \leq \int_h^0 |f'(t)|\, dt \leq \epsilon |h|.$$

   All in all, for all $h \in (-\delta, 0) \cup (0, \delta)$,

$$|f(0) - f(h)| \leq \epsilon |h|,$$

   that is

$$\left| \frac{f(0) - f(h)}{h} \right| \leq \epsilon,$$

   or equivalently

$$\lim_{h \to 0} \frac{f(0) - f(h)}{h} = 0.$$

   This yields that $f$ is differentiable at 0, with $f'(0) = 0$.

4. Taylor's expansions can be helpful. For example, as $x \to 0$:

$$\begin{aligned} &\cos x = 1 + o(x), \\ &e^x = 1 + x + o(x), \\ &\sin x = x + o(x), \\ &e^{x^2} = 1 + o(x), \\ &e^{x^4} = 1 + o(x), \end{aligned}$$

that is

$$\begin{aligned}\cos x + e^x \sin x + e^{x^2} - e^{x^4} \\ &= 1 + (1 + x + o(x))(x + o(x)) + 1 - 1 + o(x) \\ &= 1 + x + o(x).\end{aligned}$$

Since

$$\ln(1+x) = x + o(x),$$

we obtain that, as $x \to 0$,

$$\begin{aligned}\left(\cos x + e^x \sin x + e^{x^2} - e^{x^4}\right)^{\frac{1}{\ln(1+x)}} \\ &= \exp\left(\frac{1}{\ln(1+x)} \ln\left(\cos x + e^x \sin x + e^{x^2} - e^{x^4}\right)\right) \\ &= \exp\left(\frac{1}{x+o(x)} \ln\left(1 + x + o(x)\right)\right) \\ &= \exp\left(\frac{1}{x+o(x)} \left(x + o(x)\right)\right) \\ &= \exp\left(\frac{1+o(1)}{1+o(1)}\right),\end{aligned}$$

yielding that

$$\lim_{x\to 0}\left(\cos x + e^x \sin x + e^{x^2} - e^{x^4}\right)^{\frac{1}{\ln(1+x)}} = e.$$

5. We suppose that $f$ satisfies (4) for some constant $M_f$ and that $g$ satisfies (4) for some constant $M_g$.

   We will show that $h$ satisfies (4) with respect to the constant

   $$M := \max\{M_f, M_g\}.$$

   To this end, let $x$, $y \in \mathbb{R}$. Up to swapping $f$ with $g$, we can assume that $f(x) \geq g(x)$ (in particular, $h(x) = f(x)$).

   Now, if $f(y) \geq g(y)$ we see that $h(y) = f(y)$, therefore

   $$|h(x) - h(y)| = |f(x) - f(y)| \leq M_f|x-y| \leq M|x-y|$$

and we are done.

Hence, we can suppose that $f(y) \le g(y)$. In this case, $h(y) = g(y)$ and

$$\begin{aligned} h(x) - h(y) = f(x) - g(y) \le f(x) - f(y) \le |f(x) - f(y)| \\ \le M_f|x - y| \le M|x - y|. \end{aligned} \tag{5}$$

In the same vein,

$$\begin{aligned} h(y) - h(x) = g(y) - f(x) \le g(x) - g(y) \le |g(x) - g(y)| \\ \le M_g|x - y| \le M|x - y|. \end{aligned} \tag{6}$$

Gathering (5) and (6), we obtain (4) for the function $h$.