## AMSI Summer School: Hyperbolic Knot Theory Quiz

**Problem 1** (Algebra)**.** Let $\mathrm{SL}(2,\mathbb{C})$ denote the set of $2 \times 2$ matrices with complex coefficients and determinant one. That is, elements have the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d$ lie in $\mathbb{C}$ and $ad - bc = 1$.

(a) Prove that the product of two matrices in $\mathrm{SL}(2,\mathbb{C})$ is in $\mathrm{SL}(2,\mathbb{C})$. Prove that the inverse of a matrix in $\mathrm{SL}(2,\mathbb{C})$ is in $\mathrm{SL}(2,\mathbb{C})$.

(b) $\mathrm{SL}(2,\mathbb{C})$ acts on $z \in \mathbb{C} \cup \{\infty\}$ as follows:

$$Az = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \mapsto \frac{az+b}{cz+d}$$

If $z = \infty$ we take this to be $\frac{a}{c}$. Show that the action of $A$ and $-A$ is identical. Thus we will often consider $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\sim$ where $A \sim -A$.

(c) Show that an element $A$ in $\mathrm{PSL}(2,\mathbb{C})$ that is not the identity either fixes one or two points in $\mathbb{C} \cup \{\infty\}$. Prove that if it has exactly one fixed point, its trace is $\pm 2$.

**Problem 2** (Distance function)**.** Recall that a function $d\colon X \times X \to [0,\infty)$ is called a distance function if it satisfies:

(1) (Symmetry) $d(x,y) = d(y,x)$ for all $x, y \in X$

(2) (Nondegeneracy) $d(x,y) = 0$ if and only if $x = y \in X$

(3) (Triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Prove that the usual Euclidean distance function $d_E\colon \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ satisfies this definition, where $d_E$ is given by:

$$d_E((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$$

**Problem 3** (Open set)**.** Suppose $X$ admits a distance function $d\colon X \times X \to [0,\infty)$. For any $x \in X$, the ball of radius $\delta > 0$ centred at $x$ is defined to be the set:

$$B(x,\delta) = \{y \in X \mid d(x,y) < \delta\}$$

A set $V \subset X$ is said to be *open* if for every $x \in V$, there exists $\delta > 0$ such that $B(x,\delta) \subset V$. It is *closed* if its complement is open.

Prove that if $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ is a collection of open sets in $X$, then the union $\bigcup_{\alpha \in \mathcal{A}} V_\alpha$ is also open. Prove that if $\{V_i\}_{i=1}^k$ is a finite collection of open sets in $X$ then the intersection $\bigcap_{i=1}^k V_i$ is open.

**Problem 4** (Cauchy sequences)**.** Again let $X$ be a space with a distance function $d$. Recall that a *Cauchy sequence* is a sequence $\{x_n\}$ such that for any $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$.

(a) Prove that if $\{x_n\}$ is a convergent sequence then it is a Cauchy sequence.

(b) Let $X$ be the rational numbers $\mathbb{Q}$ with the usual distance function inherited from $\mathbb{R}$. Show that there exists a Cauchy sequence in $X$ that does not converge in $X$.

**Problem 5** (Homotopy of curves)**.** A *curve* in a space $X$ is a continuous map $\gamma\colon [0,1] \to X$.

Suppose $\gamma_0$ and $\gamma_1$ are curves, and suppose $\gamma_0(0) = \gamma_1(0)$, and $\gamma_0(1) = \gamma_1(1)$. We say the two curves $\gamma_0$ and $\gamma_1$ are *homotopic* if there exists a continuous map $H\colon [0,1] \times [0,1] \to X$ such that $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$. They are *homotopic rel endpoints* if in addition, $H(0,s) = \gamma_0(0) = \gamma_1(0)$ and $H(1,s) = \gamma_0(1) = \gamma_1(1)$. The map $H$ is called a *homotopy.* These notions give relations on the set of curves in $X$, namely homotopy of curves and homotopy of curves rel endpoints.

Prove that homotopy of curves is an equivalence relation. Similarly, homotopy of curves rel endpoints is an equivalence relation.

*Solution.*[Problem 1 Algebra] (a) The product of two matrices in $\mathrm{SL}(2,\mathbb{C})$ and the inverse are given as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

All entries are in $\mathbb{C}$. The determinant of the product is

$$(ae+bg)*(cf+dh)-(af+bh)(ce+dg) = (ad-bc)(ef-gh) = 1$$

The determinant of the inverse is $ad-bc=1$.

(b) Note $-A$ adjusts the matrix $A$ by multiplying all entries by $-1$. For the action on $\mathbb{C}\cup\{\infty\}$, this has the effect of multiplying both entries in the numerator by $-1$ and both entries in the denominator by $-1$. Hence they cancel, and the action agrees with that of $A$.

(c) We find the fixed points of the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{C}\cup\{\infty\}$. Suppose

$$\frac{az+b}{cz+d} = z. \text{ Then } az+b = cz^2+dz$$

Thus $z$ is a root of the polynomial $cz^2+(d-a)z-b$ with complex coefficients. The roots are given by the quadratic formula:

$$z = \frac{(a-d)\pm\sqrt{(d-a)^2+4cb}}{2c}$$

There will be two roots, corresponding to two fixed points, unless the discriminant $(d-a)^2+4bc$ is zero, in which case there will be one root.

If the discriminant is zero, we have $d^2-2da+a^2+4bc=0$. Note the left hand side of this equation is $d^2+a^2-4(ad-bc)+2ad$, or $(d+a)^2-4$. So there is one root if and only if $(d+a)^2=4$, or $d+a=\pm 2$.

*Solution.*[Problem 2 Distance function] Symmetry follows from the fact that

$$(x_i-y_i)^2 = (y_i-x_i)^2$$

so $d(x,y)=d(y,x)$.

We have $d(x,y)=0$ if and only if $\sum(x_i-y_i)^2=0$. Since each term in the sum is nonnegative, this is zero if and only if each $(x_i-y_i)=0$, or if and only if $(x_1,\dots,x_n)=(y_1,\dots,y_n)$.

For the triangle inequality, for each $i$ we have

$$(x_i-z_i)^2 = (x_i-y_i+y_i-z_i)^2 \le (x_i-y_i)^2+(y_i-z_i)^2$$

*Solution.*[Problem 3 Open set] To show the union is open, we need to show that for every point $x$ in the union, there is $\delta>0$ such that the ball $B(x,\delta)$ lies inside the union. But if $x$ is in the union, then $x$ must be in one of the $V_\alpha$. Because $V_\alpha$ is open, there exists $\delta_\alpha>0$ such that $B(x,\delta_\alpha)\subset V_\alpha$. Since $V_\alpha\subset\bigcup_\beta V_\beta$, the ball $B(x,\delta_\alpha)$ also lies in the union.

For the intersection, suppose $x\in\bigcap V_i$. Then $x\in V_i$ for each $i$. Thus there exist $\delta_1,\dots,\delta_k$ such that each $\delta_i>0$ and $B(x,\delta_i)\subset V_i$. Let $\delta$ be the minimum of the $\delta_i$. Then $\delta>0$ and $B(x,\delta)\subset B(x,\delta_i)\subset V_i$ for $i=1,\dots,k$. Thus $B(x,\delta)\subset\bigcap V_i$.

*Solution.*[Problem 4 Cauchy sequences] (a) Fix $\epsilon>0$. The fact that $\{x_n\}$ is a convergent sequence means that there exists $x\in X$ such that for any $\epsilon>0$, there exists an integer $N$ such that $n \geq N$ implies $x_n \in B(x, \epsilon/2)$. Then for $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon.$$

Here the first inequality is by the triangle inequality, and the second follows from the fact that $x_n$ and $x_m$ lie in $B(x, \epsilon/2)$.

(b) Choose a sequence of rational numbers converging to an irrational number. For example, take $\{x_n\} \subset \mathbb{Q}$ converging to $\sqrt{2}$ in $\mathbb{R}$. Then for any $\epsilon > 0$, as above there exists $N$ such that $n, m \geq N$ implies $d(x_n, x_m) \leq d(x_n, \sqrt{2}) + d(\sqrt{2}, x_m) < \epsilon$, so this is a Cauchy sequence. However, it cannot converge to any point in $\mathbb{Q}$. For if $y \in \mathbb{Q}$, then there exists $\delta > 0$ such that $d(y, \sqrt{2}) > \delta/2$. Then for sufficiently large $N$, $x_n$ lies in $B(\sqrt{2}, \delta/2)$, which is disjoint from the ball $B(y, \delta/2)$. Thus $\{x_n\}$ cannot converge to $y \in \mathbb{Q}$.

*Solution.*[Problem 5 Homotopy of curves] We need to show homotopy is reflexive, symmetric, and transitive.

Reflexive: For $\gamma\colon [0,1] \to X$, we define $H\colon [0,1] \times [0,1] \to X$ to be $H(t,s) = \gamma(t)$. Then this is continuous because $\gamma$ is continuous, and $H(t,0) = H(t,1) = \gamma(t)$. Note the same argument applies for homotopy rel endpoints.

Symmetric: If $\gamma_0$ is homotopic to $\gamma_1$, then there exists a continuous map $H\colon [0,1] \times [0,1] \to X$ with $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$. Then define $\overline{H}\colon [0,1] \times [0,1] \to X$ by $\overline{H}(t,s) = H(t, 1-s)$. Then $\overline{H}(t,0) = H(t,1) = \gamma_1(t)$ and $\overline{H}(t,1) = H(t,0) = \gamma_0(t)$. Thus homotopy is symmetric. Observe that if $H(0,s) = \gamma_0(0) = \gamma_1(0)$ for all $s$, then $\overline{H}(0,s) = \gamma_0(s) = \gamma_1(s)$ for all $s$, and similarly for $\overline{H}(1,s)$. So symmetry holds rel endpoints.

Transitive: Suppose $\gamma_0$ is homotopic to $\gamma_1$ and $\gamma_1$ is homotopic to $\gamma_2$, via homotopies $H$ and $G$. Then let $F\colon [0,1] \times [0,1] \to X$ be defined by

$$F(t,s) = \begin{cases} H(t, 2s) & \text{if } 0 \leq s \leq 1/2 \\ G(t, 2s-1) & \text{if } 1/2 \leq s \leq 1 \end{cases}$$

Note when $s = 1/2$, $H(t,2s) = H(t,1) = \gamma_1(t) = G(t,0) = G(t, 2s-1)$, so $F$ is continuous. Then $F(t,0) = H(t,0) = \gamma_0(t)$ and $F(t,1) = G(t,1) = \gamma_2(t)$, so $F$ is a homotopy from $\gamma_0$ to $\gamma_2$. Thus it is transitive.

If the homotopies $H$ and $G$ are rel endpoints, then so is the homotopy $F$. So homotopy rel endpoints is also transitive.