High-Order Numerical Methods for Time-Dependent PDEs

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Practice questions

The following questions will help you prepare and assess your readiness for the course.

Let $u, v \in C^{\infty}(\Omega)$, and define the standard L_2-scalar product and norm

$$(v, u) = \int_{\Omega} uv \, dx, \quad \|u\|^2 = (u, u).$$

1. Let $\Omega = [0, 1]$. Show that

$$\left(u, \frac{dv}{dx}\right) + \left(\frac{du}{dx}, v\right) = u(1)v(1) - u(0)v(0).$$

Discretise the interval $\Omega = [0, 1]$ uniformly into n grid points with $x_j = (j-1)\Delta x$, $j = 1, 2, \ldots, n$, $\Delta x = 1/(n-1)$. Denote $u_j = u(x_j)$ with $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ being the restriction of the smooth function $u : \mathbb{R} \to \mathbb{R}$ on the grid x_j. We introduce the diagonal matrix operators $\mathbf{I}_h = \Delta x \, \mathrm{diag}([1, 1, \cdots, 1])$ and $\mathbf{H} = \Delta x \, \mathrm{diag}([h_1, h_2, \cdots, h_n])$, where $h_j > 0$ are real positive weights independent of Δx. Define the discrete scalar products

$$\langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{H}} := \mathbf{v}^T \mathbf{H} \mathbf{u} = \Delta x \sum_{j=1}^{n} u_j v_j h_j, \quad \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{I}_h} := \mathbf{v}^T \mathbf{I}_h \mathbf{u} = \Delta x \sum_{j=1}^{n} u_j v_j. \tag{1}$$

2. Show that the discrete norms $\|\mathbf{u}\|^2_{\mathbf{I}_h} = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{I}_h}$, $\|\mathbf{u}\|^2_{\mathbf{H}} = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{H}}$, are equivalent.

Exactness of quadrature rules for polynomials. Let $\mathbf{1} := (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, then

$$\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} \approx \int_0^1 u \, dx. \tag{2}$$

3. For a monomial $u(x) = x^p$ with $p \in \mathbb{N}$. Show that if $h_1 = h_n = 1/2$ and $h_j = 1$, for $1 < j < n$ and $p = 1$, then $\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \int_0^1 u \, dx$ for all $n \geq 2$.

[Think about: Can we construct composite quadrature rules such that $\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \int_0^1 u \, dx$ for finite n and fixed $p > 1$?]

4. Let $\mathbf{H}$ be defined by $h_1 = h_n = 17/48$, $h_2 = h_{n-1} = 59/48$, $h_3 = h_{n-2} = 43/48$, $h_4 = h_{n-3} = 49/48$, and $h_j = 1$, for $4 < j < n-3$. Write a simple Python code to verify that $\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \int_0^1 u \, dx$ for $n \geq 8$, with $u(x) = x^p$ and $0 \leq p \leq 3$.

Summation-by-parts principle. Let $D \in \mathbb{R}^{n \times n}$ denote a discrete derivative operator on the grid, that is $(D\mathbf{u})_j \approx \partial u / \partial x|_{x=x_j}$.

5. Show that if $D \in \mathbb{R}^{n \times n}$ satisfies

$$D = \mathbf{H}^{-1} Q, \quad Q + Q^T = B := \mathrm{diag}([-1, 0, \cdots, 1]), \quad \mathbf{H} = \mathbf{H}^T > 0, \tag{3}$$

then

$$\langle \mathbf{v}, D\mathbf{u} \rangle_{\mathbf{H}} + \langle D\mathbf{v}, \mathbf{u} \rangle_{\mathbf{H}} = u_n v_n - u_1 v_1.$$

Let $D \in \mathbb{R}^{n\times n}$ be defined by

$$(D\mathbf{u})_j = \left\{ \begin{array}{cc} \frac{u_2-u_1}{\Delta x}, & j=1 \\ \frac{u_{j+1}-u_{j-1}}{2\Delta x}, & 1<j<n \\ \frac{u_n-u_{n-1}}{\Delta x}, & j=n \end{array} \right\}. \tag{4}$$

6. Show that for a sufficiently smooth function $u : \mathbb{R} \to \mathbb{R}$ the discrete derivative operator $(D\mathbf{u})_j$ defined in (4) can be written as

$$(D\mathbf{u})_j = \frac{du}{dx}|_{x=x_j} + \mathbb{T}_j, \tag{5}$$

where $\mathbb{T}$ is the truncation error. Determine the truncation error $\mathbb{T}_j$ for all $j = 1, 2, \ldots n$. Discuss what happens to the truncation error as $\Delta x \to 0$.

7. Write a simple Python code implementing the discrete derivative operator $(D\mathbf{u})_j$ defined in (4). Consider $u(x) = x^3$ for $x \in [0, 1]$ and verify the accuracy of the operator, compare the error $e_j = |(D\mathbf{u})_j - u'(x_j)|$ to the truncation error $\mathbb{T}_j$.

8. Show that the discrete derivative operator $(D\mathbf{u})_j$ defined in (4) satisfies (3), that is

$$D = \mathbf{H}^{-1}Q, \quad Q + Q^T = B := \mathrm{diag}([-1, 0, \cdots, 1]), \quad \mathbf{H} = \mathbf{H}^T > 0,$$

where $h_1 = h_n = 1/2$ and $h_j = 1$, for $1 < j < n$. Determine the corresponding matrix operator Q.

9. Consider the IVP

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},\ x \in (-\infty, \infty),\ t \geq 0,\ a > 0, \tag{6a}$$

$$u(x, 0) = f(x), \tag{6b}$$

Verify that $u(x, t) = f(x - at)$ solves the IVP.

10. Consider the IBVP

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},\ x \in (0, \infty),\ t \geq 0,\ a > 0, \tag{7a}$$

$$u(x, 0) = 0,\ x \in \Omega, \tag{7b}$$

$$u(0, t) = g(t),\ t \geq 0, \tag{7c}$$

with compatible data, that is $g(0) = u(0, 0) = 0$. Show that

$$u(x,t) = \left\{ \begin{array}{cc} g(t - x/a), & \text{if } t - x/a \geq 0 \\ 0, & \text{else} \end{array} \right\}.$$

solves the IBVP.

11. Consider the linear differential operator

$$P\left(\frac{\partial}{\partial x}\right)u = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu, \quad x \in \mathbb{R},$$

for real constants $a > 0,\ b,\ c \in \mathbb{R}$ with the decay condition $|u| \to 0$ at $|x| \to \infty$. Show that the operator P is semi-bounded, that is $(u, Pu) \leq \alpha_c\|u\|^2$ for some $\alpha_c \in \mathbb{R}$.

12. Consider the IVP

$$\frac{\partial u}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u, \quad P\left(\frac{\partial}{\partial x}\right)u = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu,\ x \in (-\infty, \infty),\ t \geq 0, \tag{8a}$$

$$u(x, 0) = e^{ikx}, \quad k \in \mathbb{R}. \tag{8b}$$

Determine $\omega \in \mathbb{C}$ so that $u(x, t) = e^{\omega t + ikx}$ is a solution to the IVP.

13. Consider the IVP

$$\frac{\partial u}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u,\ x \in (-\infty, \infty),\ t \geq 0, \tag{9a}$$

$$u(x, 0) = f(x). \tag{9b}$$

Show that if the differential operator P is semi-bounded, that is $(u, Pu) \leq \alpha_c \|u\|^2$, then

$$\|u(t)\| \leq e^{\alpha_c t}\|f\|, \quad \forall\, t \geq 0.$$

14. Consider the semi-discrete approximation of the IVP

$$\frac{d\mathbf{u}}{dt} = \mathcal{P}\mathbf{u}, \quad t \geq 0, \quad \mathbf{u}(t) \in \mathbb{R}^n, \tag{10a}$$

$$\mathbf{u}(0) = \mathbf{f}, \tag{10b}$$

where $\mathcal{P} \approx P\left(\partial/\partial x\right)$ approximates the spatial differential operator on a grid and the time derivative is left continuous. Show that the if the discrete spatial differential operator $\mathcal{P}$ is semi-bounded in a discrete scalar product, that is $\langle \mathbf{u}, \mathcal{P}\mathbf{u}\rangle_{\mathbf{H}} \leq \alpha_d \|\mathbf{u}\|^2_{\mathbf{H}}$ then

$$\|\mathbf{u}(t)\|_{\mathbf{H}} \leq e^{\alpha_d t}\|\mathbf{f}\|_{\mathbf{H}}, \quad \forall\, t \geq 0.$$

15. Let $x_0, x_1, x_2 \in \mathbb{R}$, $x_0 \neq x_2$. Show that there exists a unique cubic polynomial p such that

$$p(x_0) = f(x_0), \quad p'(x_1) = f'(x_1), \quad p''(x_1) = f''(x_1), \quad p(x_2) = f(x_2),$$

where f is a given function.

16. Let us consider the first order ODE

$$\frac{dy}{dt} = f(t, y), \quad t \geq 0,$$
$$y(0) = c, \quad c \in \mathbb{R}.$$

In order to numerically solve this ODE, we introduce a discretisation $t_{n+1} = t_n + h$, where $h > 0$ is the step size, $t_0 = 0$, $n = 1, 2, \ldots$. We compute the numerical approximation for this ODE using the following recurrence formula:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_{n+1}, y_n + hf(t_n, y_n)) + f(t_n, y_n)\right), \quad y_0 = c, \quad n = 1, 2, \ldots. \tag{11}$$

Here, y_n is the numerical approximation of the solution y at $t = t_n$, that is $y_n \approx y(t_n)$.

- Suppose that $f(t, y) = \lambda y$, where λ is a complex constant such that $\mathrm{Re}(\lambda) < 0$. Show that for all $n \geq 1$, we have
$$y_n = \rho^n y_0,$$
where $\rho = 1 + h\lambda + \frac{h^2\lambda^2}{2}$. Using this result, derive a condition on the step size h such that the numerical approximation (11) is stable, that is $|y_n| < \infty$ for all $n \geq 0$.
- Write a code to implement (11). Test your code with $f(t, y) = \lambda y$, where $y_0 = 10$ and $\lambda = -1$ is a constant. Run your code until the final time $t = 10$, with various step sizes and compare your results with the analytical solution $y(t) = y_0 e^{\lambda t}$. Try to see what happens if the step size violates the condition that you have derived previously.

17. Consider the iteration equation

$$\mathbf{u}^{k+1} = A\mathbf{u}^k, \quad \mathbf{u}^0 = \mathbf{f}, \quad k \geq 0, \quad \mathbf{u}^k \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}.$$

What is the necessary condition on the iteration matrix A so that if $\|\mathbf{f}\| < \infty$ then $\|\mathbf{u}^k\| < \infty$ for $k \to \infty$?

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1. Let $\Omega = [0, 1]$. Show that

$$\left(u, \frac{dv}{dx}\right) + \left(\frac{du}{dx}, v\right) = u(1)v(1) - u(0)v(0).$$

Sol: Applying the integration-by-parts formula to the first term gives

$$\begin{aligned}\left(u, \frac{dv}{dx}\right) + \left(\frac{du}{dx}, v\right) &= \int_0^1 u\frac{dv}{dx} \, dx + \int_0^1 \frac{du}{dx} v \, dx \\ &= uv|_0^1 - \int_0^1 \frac{du}{dx} v \, dx + \int_0^1 \frac{du}{dx} v \, dx \\ &= u(1)v(1) - u(0)v(0).\end{aligned}$$

Discretise the interval $\Omega = [0, 1]$ uniformly into n grid points with $x_j = (j-1)\Delta x$, $j = 1, 2, \dots, n$, $\Delta x = 1/(n-1)$. Denote $u_j = u(x_j)$ with $\mathbf{u} = (u_1, u_2, \dots, u_n)^T \in \mathbb{R}^n$ being the restriction of the smooth function $u : \mathbb{R} \to \mathbb{R}$ on the grid x_j. We introduce the diagonal matrix operators $\mathbf{I}_h = \Delta x \, \mathrm{diag}([1, 1, \cdots, 1])$ and $\mathbf{H} = \Delta x \, \mathrm{diag}([h_1, h_2, \cdots, h_n])$, where $h_j > 0$ are real positive weights independent of Δx. Define the discrete scalar products

$$\langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{H}} := \mathbf{v}^T \mathbf{H} \mathbf{u} = \Delta x \sum_{j=1}^{n} u_j v_j h_j, \quad \langle \mathbf{v}, \mathbf{u} \rangle_{\mathbf{I}_h} := \mathbf{v}^T \mathbf{I}_h \mathbf{u} = \Delta x \sum_{j=1}^{n} u_j v_j. \tag{1}$$

2. Show that the discrete norms $\|\mathbf{u}\|_{\mathbf{I}_h}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{I}_h}$, $\|\mathbf{u}\|_{\mathbf{H}}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{H}}$, are equivalent.

Sol: These two discrete norms are equivalent if there exists two positive real constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|\mathbf{u}\|_{\mathbf{I}_h} \le \|\mathbf{u}\|_{\mathbf{H}} \le \beta \|\mathbf{u}\|_{\mathbf{I}_h}. \tag{2}$$

From the definition of the discrete norms, we have

$$\|\mathbf{u}\|_{\mathbf{H}}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{H}} = \Delta x \sum_{j=1}^{n} h_j u_j^2 \le \left(\max_i h_i\right) \Delta x \sum_{j=1}^{n} u_j^2 = \left(\max_i h_i\right) \|\mathbf{u}\|_{\mathbf{I}_h}^2. \tag{3}$$

Similarly,

$$\|\mathbf{u}\|_{\mathbf{H}}^2 \ge \left(\min_i h_i\right) \|\mathbf{u}\|_{\mathbf{I}_h}^2. \tag{4}$$

Therefore,

$$\left(\min_i h_i\right) \|\mathbf{u}\|^2_{\mathbf{I}_h} \leq \|\mathbf{u}\|^2_{\mathbf{H}} \leq \left(\max_i h_i\right) \|\mathbf{u}\|^2_{\mathbf{I}_h}, \tag{5}$$

and hence the norms are equivalent with $\alpha = \sqrt{(\min_i h_i)} > 0$ and $\beta = \sqrt{(\max_i h_i)} > 0$.

Exactness of quadrature rules for polynomials. Let $\mathbf{1} := (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, then

$$\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} \approx \int_0^1 u\, dx. \tag{6}$$

3. For a monomial $u(x) = x^p$ with $p \in \mathbb{N}$. Show that if $h_1 = h_n = 1/2$ and $h_j = 1$, for $1 < j < n$ and $p = 1$, then $\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \int_0^1 u\, dx$ for all $n \geq 2$.

Sol: From the definition of the discrete scalar products, we have

$$\begin{aligned}
\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \mathbf{1}^T \mathbf{H} \mathbf{u} = \sum_{j=1}^{n} \Delta x h_j u_j &= \Delta x \sum_{j=1}^{n} h_j u(x_j) = \Delta x \sum_{j=1}^{n} h_j x_j \\
&= \Delta x^2 \sum_{j=1}^{n} (j-1) h_j = \Delta x^2 \left(\sum_{j=2}^{n-1} (j-1) + \frac{1}{2}(n-1) \right) \\
&= \Delta x^2 \left(\sum_{j=1}^{n-2} j + \frac{1}{2}(n-1) \right) = \Delta x^2 \left(\frac{1}{2}(n-1)(n-2) + \frac{1}{2}(n-1) \right) \\
&= \frac{1}{2} \Delta x^2 (n-1)^2 = \frac{1}{2} = \int_0^1 u\, dx.
\end{aligned}$$

[Think about: Can we construct composite quadrature rules such that $\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \int_0^1 u\, dx$ for finite n and fixed $p > 1$?]

4. Let $\mathbf{H}$ be defined by $h_1 = h_n = 17/48$, $h_2 = h_{n-1} = 59/48$, $h_3 = h_{n-2} = 43/48$, $h_4 = h_{n-3} = 49/48$, and $h_j = 1$, for $4 < j < n-3$. Write a simple Python code to verify that $\langle \mathbf{1}, \mathbf{u} \rangle_{\mathbf{H}} = \int_0^1 u\, dx$ for $n \geq 8$, with $u(x) = x^p$ and $0 \leq p \leq 3$.

Sol: See attached Jupyter Notebook

Summation-by-parts principle. Let $D \in \mathbb{R}^{n \times n}$ denote a discrete derivative operator on the grid, that is $(D\mathbf{u})_j \approx \partial u / \partial x|_{x=x_j}$.

5. Show that if $D \in \mathbb{R}^{n \times n}$ satisfies

$$D = \mathbf{H}^{-1} Q, \quad Q + Q^T = B := \text{diag}([-1, 0, \cdots, 1]), \quad \mathbf{H} = \mathbf{H}^T > 0, \tag{7}$$

then

$$\langle \mathbf{v}, D\mathbf{u} \rangle_{\mathbf{H}} + \langle D\mathbf{v}, \mathbf{u} \rangle_{\mathbf{H}} = u_n v_n - u_1 v_1.$$

Sol: Consider

$$\langle \mathbf{v}, D\mathbf{u} \rangle_{\mathbf{H}} + \langle D\mathbf{v}, \mathbf{u} \rangle_{\mathbf{H}} = \mathbf{v}^T (\mathbf{H} D) \mathbf{u} + \mathbf{v}^T (\mathbf{H} D)^T \mathbf{u}.$$

Note that $\mathbf{H} D = Q$ and $Q + Q^T = B := \text{diag}([-1, 0, \cdots, 1])$, then

$$\langle \mathbf{v}, D\mathbf{u} \rangle_{\mathbf{H}} + \langle D\mathbf{v}, \mathbf{u} \rangle_{\mathbf{H}} = \mathbf{v}^T (Q + Q^T) \mathbf{u} = \mathbf{v}^T B \mathbf{u} = u_n v_n - u_1 v_1.$$

Let $D \in \mathbb{R}^{n \times n}$ be defined by

$$(D\mathbf{u})_j = \left\{ \begin{array}{ll} \frac{u_2 - u_1}{\Delta x}, & j = 1 \\ \frac{u_{j+1} - u_{j-1}}{2\Delta x}, & 1 < j < n \\ \frac{u_n - u_{n-1}}{\Delta x}, & j = n \end{array} \right\}. \tag{8}$$

6. Show that for a sufficiently smooth function $u : \mathbb{R} \to \mathbb{R}$ the discrete derivative operator $(D\mathbf{u})_j$ defined in (8) can be written as

$$(D\mathbf{u})_j = \frac{du}{dx}|_{x=x_j} + \mathbb{T}_j, \tag{9}$$

where $\mathbb{T}$ is the truncation error. Determine the truncation error $\mathbb{T}_j$ for all $j = 1, 2, \ldots n$. Discuss what happens to the truncation error as $\Delta x \to 0$.

Sol: We use Taylor expansions

$$u(x+\Delta x) = u(x) + \Delta x \frac{du}{dx} + \frac{\Delta x^2}{2}\frac{d^2u(x)}{dx^2} + \frac{\Delta x^3}{6}\frac{d^3u(x)}{dx^3} + \cdots,$$
$$u(x-\Delta x) = u(x) - \Delta x \frac{du}{dx} + \frac{\Delta x^2}{2}\frac{d^2u(x)}{dx^2} - \frac{\Delta x^3}{6}\frac{d^3u(x)}{dx^3} + \cdots,$$

then we have

$$\frac{u(x+\Delta x)-u(x)}{\Delta x} = \frac{du(x)}{dx} + \frac{\Delta x}{2}\frac{d^2u(x)}{dx^2} + \frac{\Delta x^2}{6}\frac{d^3u(x)}{dx^3} + \cdots,$$
$$\frac{u(x+\Delta x)-u(x-\Delta x)}{2\Delta x} = \frac{du(x)}{dx} + \frac{\Delta x^2}{6}\frac{d^3u(x)}{dx^3} + \cdots,$$
$$\frac{u(x)-u(x-\Delta x)}{\Delta x} = \frac{du}{dx} - \frac{\Delta x}{2}\frac{d^2u(x)}{dx^2} + \frac{\Delta x^2}{6}\frac{d^3u(x)}{dx^3} + \cdots.$$

Taylor remainder theorem gives

$$(D\mathbf{u})_j = \left\{ \begin{array}{ll} \frac{u_2-u_1}{\Delta x} & = \frac{du(x_1)}{dx} + \mathbb{T}_1,\ j=1 \\ \frac{u_{j+1}-u_{j-1}}{2\Delta x} & = \frac{du(x_j)}{dx} + \mathbb{T}_j, 1<j<n \\ \frac{u_n-u_{n-1}}{\Delta x} & = \frac{du(x_n)}{dx} + \mathbb{T}_n,\ j=n \end{array} \right\},$$

where

$$\mathbb{T}_j = \left\{ \begin{array}{l} \frac{\Delta x}{2}\frac{d^2u(\xi)}{dx^2},\ x_1 \le \xi \le x_2,\ j=1 \\ \frac{\Delta x^2}{6}\frac{d^3u(\xi)}{dx^3} + \cdots, 1<j<n,\ x_{j-1} \le \xi \le x_{j+1} \\ -\frac{\Delta x}{2}\frac{d^2u(\xi)}{dx^2},\ x_{n-1} \le \xi \le x_n,\ j=n \end{array} \right\}.$$

Note that $\mathbb{T}_1 = O(\Delta x)$, $\mathbb{T}_n = O(\Delta x)$, and $\mathbb{T}_j = O(\Delta x^2)$ for $1 < j < n$. Thus we must have $\Delta x \to 0 \implies \mathbb{T}_j \to 0$ for all $j = 1, 2, \cdots, n$.

7. Write a simple Python code implementing the discrete derivative operator $(D\mathbf{u})_j$ defined in (8). Consider $u(x) = x^3$ for $x \in [0,1]$ and verify the accuracy of the operator, compare the error $e_j = |(D\mathbf{u})_j - u'(x_j)|$ to the truncation error $\mathbb{T}_j$.

Sol: See attached Jupyter Notebook

8. Show that the discrete derivative operator $(D\mathbf{u})_j$ defined in (8) satisfies (7), that is

$$D = \mathbf{H}^{-1}Q, \quad Q + Q^T = B := \text{diag}([-1, 0, \cdots, 1]), \quad \mathbf{H} = \mathbf{H}^T > 0,$$

where $h_1 = h_n = 1/2$ and $h_j = 1$, for $1 < j < n$. Determine the corresponding matrix operator Q.

Sol: For simplicity, we consider $n = 5$. The matrix for the derivative operator D is given by

$$D = \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} = \mathbf{H}^{-1} \underbrace{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{Q}.$$

Then we have

$$Q + Q^T = B = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag}([-1, 0, \cdots, 1]).$$

9. Consider the IVP

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},\ x \in (-\infty,\infty),\ t \geq 0,\ a > 0, \tag{10a}$$

$$u(x,0) = f(x), \tag{10b}$$

Verify that $u(x,t) = f(x - at)$ solves the IVP.

Sol: Clearly, we have $u(x,0) = f(x - a(0)) = f(x)$, which implies that the initial condition is satisfied. To show that u satisfies the PDE, we compute the first partial derivatives of u. We obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}f(x-at) = -af'(x-at)$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(x-at) = f'(x-at).$$

Hence

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -af'(x-at) + af'(x-at) = 0.$$

Therefore, u solves the IVP.

10. Consider the IBVP

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x},\ x \in (0,\infty),\ t \geq 0,\ a > 0, \tag{11a}$$

$$u(x,0) = 0,\ x \in \Omega, \tag{11b}$$

$$u(0,t) = g(t),\ t \geq 0, \tag{11c}$$

with compatible data, that is $g(0) = u(0,0) = 0$. Show that

$$u(x,t) = \left\{ \begin{array}{rl} g(t-x/a), & \text{if } t - x/a \geq 0 \\ 0, & \text{else} \end{array} \right\}.$$

solves the IBVP.

Sol: When $x = 0$, we have $t - x/a = t \geq 0$, and hence $u(0,t) = g(t)$ for all $t \geq 0$. When $t = 0$, it is clear that $t - x/a = -x/a < 0$, which implies $u(x,0) = 0$. Therefore, u satisfies the initial and boundary conditions. To show that u also satisfies the PDE, we compute the first derivatives

$$\frac{\partial u}{\partial t} = \left\{ \begin{array}{rl} g'(t-x/a), & \text{if } t - x/a \geq 0 \\ 0, & \text{else} \end{array} \right\},$$

$$\frac{\partial u}{\partial x} = \left\{ \begin{array}{rl} -\frac{1}{a}g'(t-x/a), & \text{if } t - x/a \geq 0 \\ 0, & \text{else} \end{array} \right\}.$$

We obtain

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \left\{ \begin{array}{rl} g'(t-x/a) - g'(t-x/a) = 0, & \text{if } t - x/a \geq 0 \\ 0, & \text{else} \end{array} \right\} = 0.$$

Thus, u solves the IBVP.

11. Consider the linear differential operator

$$P\left(\frac{\partial}{\partial x}\right)u = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu, \quad x \in \mathbb{R},$$

for real constants $a > 0,\ b,\ c \in \mathbb{R}$ with the decay condition $|u| \to 0$ at $|x| \to \infty$. Show that the operator P is semi-bounded, that is $(u, Pu) \leq \alpha_c\|u\|^2$ for some $\alpha_c \in \mathbb{R}$.

Sol: We consider (u, Pu) and use integration by parts, we have

$$\begin{aligned}(u,Pu) &= a\left(u, \frac{\partial^2 u}{\partial x^2}\right) + b\left(u, \frac{\partial u}{\partial x}\right) + (u, cu) \\ &= -a\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) + (u, cu) + au\frac{\partial u}{\partial x}\Big|_{x=-\infty}^{x=\infty} + b\frac{u^2}{2}\Big|_{x=-\infty}^{x=\infty}.\end{aligned}$$

Using the decay conditions to eliminate the boundary terms yields

$$(u, Pu) = -a\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right) + c\,(u,u) \leq c\,(u,u) = c\|u\|^2.$$

Therefore

$$(u, Pu) \leq \alpha_c \|u\|^2, \quad \alpha_c = c.$$

12. Consider the IVP

$$\frac{\partial u}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u, \quad P\left(\frac{\partial}{\partial x}\right)u = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu, \; x \in (-\infty, \infty), \; t \geq 0, \tag{12a}$$

$$u(x,0) = e^{ikx}, \quad k \in \mathbb{R}. \tag{12b}$$

Determine $\omega \in \mathbb{C}$ so that $u(x,t) = e^{\omega t + ikx}$ is a solution to the IVP.

Sol: Inserting $u(x,t) = e^{\omega t + ikx}$ in (12) we have

$$\omega u = \widehat{P}(ik)u, \quad \text{where } \widehat{P}(ik) = -ak^2 + ibk + c.$$

To ensure that u satisfies the PDE, We must have $\omega = \widehat{P}(ik)$. Note that $u(x,0) = e^{ikx}$ satisfies the initial condition.

13. Consider the IVP

$$\frac{\partial u}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u, \; x \in (-\infty, \infty), \; t \geq 0, \tag{13a}$$

$$u(x,0) = f(x). \tag{13b}$$

Show that if the differential operator P is semi-bounded, that is $(u, Pu) \leq \alpha_c \|u\|^2$, then

$$\|u(t)\| \leq e^{\alpha_c t}\|f\|, \quad \forall\, t \geq 0.$$

Sol: We consider

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = \left(u, \frac{\partial u}{\partial t}\right) = (u, Pu) \leq \alpha_c \|u\|^2,$$

and we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 \leq \alpha_c\|u\|^2 \iff \frac{d}{dt}\|u\| \leq \alpha_c \|u\|.$$

It follows that we have the differential inequality

$$\frac{d}{dt}\|u\| - \alpha_c\|u\| \leq 0.$$

We recognize that the left hand side of the inequality is in the form of a first order linear ODE, and hence we multiply both sides of the inequality by the integrating factor $e^{-\alpha_c t}$. We obtain

$$e^{-\alpha_c t}\frac{d}{dt}\|u\| - e^{-\alpha_c t}\alpha_c\|u\| = \frac{d}{dt}\left(e^{-\alpha_c t}\|u\|\right) \leq 0,$$

where we have used the product rule. Integrating both sides from 0 to T gives

$$e^{-\alpha_c T}\|u(T)\| - \|u(0)\| \leq 0,$$

which implies

$$\|u(T)\| \leq e^{\alpha_c T}\|f\|.$$

Since $T \geq 0$ is arbitrary, we conclude that $\|u(t)\| \leq e^{\alpha_c t}\|f\|$ for all $t \geq 0$.

14. Consider the semi-discrete approximation of the IVP

$$\frac{d\mathbf{u}}{dt} = \mathcal{P}\mathbf{u}, \quad t \geq 0, \quad \mathbf{u}(t) \in \mathbb{R}^n, \tag{14a}$$

$$\mathbf{u}(0) = \mathbf{f}, \tag{14b}$$

where $\mathcal{P} \approx P\left(\partial/\partial x\right)$ approximates the spatial differential operator on a grid and the time derivative is left continuous. Show that the if the discrete spatial differential operator $\mathcal{P}$ is semi-bounded in a discrete scalar product, that is $\langle \mathbf{u}, \mathcal{P}\mathbf{u}\rangle_{\mathbf{H}} \leq \alpha_d \|\mathbf{u}\|_{\mathbf{H}}^2$ then

$$\|\mathbf{u}(t)\|_{\mathbf{H}} \leq e^{\alpha_d t}\|\mathbf{f}\|_{\mathbf{H}}, \quad \forall\, t \geq 0.$$

Sol: The steps are similar as above, the only difference here is that we consider a discrete scalar product $\langle\cdot,\cdot\rangle_{\mathbf{H}}$ and a discrete norm $\|\cdot\|_{\mathbf{H}}$. Again we consider

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{\mathbf{H}}^2 = \langle u, \frac{d\mathbf{u}}{dt}\rangle_{\mathbf{H}} = \langle \mathbf{u}, \mathcal{P}\mathbf{u}\rangle_{\mathbf{H}} \leq \alpha_d\|\mathbf{u}\|_{\mathbf{H}}^2,$$

and we have

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{\mathbf{H}}^2 \leq \alpha_d\|\mathbf{u}\|_{\mathbf{H}}^2 \iff \frac{d}{dt}\|\mathbf{u}\|_{\mathbf{H}} \leq \alpha_d\|\mathbf{u}\|_{\mathbf{H}}.$$

It follows that we have the differential inequality

$$\frac{d}{dt}\|\mathbf{u}\|_{\mathbf{H}} - \alpha_d\|\mathbf{u}\|_{\mathbf{H}} \leq 0,$$

which gives

$$\|\mathbf{u}(t)\|_{\mathbf{H}} \leq e^{\alpha_d t}\|\mathbf{f}\|_{\mathbf{H}}.$$

15. Let $x_0, x_1, x_2 \in \mathbb{R}$, $x_0 \neq x_2$. Show that there exists a unique cubic polynomial p such that

$$p(x_0) = f(x_0), \quad p'(x_1) = f'(x_1), \quad p''(x_1) = f''(x_1), \quad p(x_2) = f(x_2),$$

where f is a given function.

Sol: Write the cubic polynomial as $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ are to be determined. The first and second derivatives of p are

$$p'(x) = 3ax^2 + 2bx + c \quad \text{and} \quad p''(x) = 6ax + 2b$$

respectively. Hence, the given conditions can be written as the following linear system:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_2) \\ f'(x_1) \\ f''(x_1) \end{bmatrix}.$$

We need to show that this linear system has a unique solution. To this end, we compute the determinant

$$\det\left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & x_2 - x_0 & x_2^2 - x_0^2 & x_2^3 - x_0^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} x_2 - x_0 & x_2^2 - x_0^2 & x_2^3 - x_0^3 \\ 1 & 2x_1 & 3x_1^2 \\ 0 & 2 & 6x_1 \end{bmatrix}\right)$$

where we have used row operations and expanded along the first column. We expand along the first column again, and we use the fact that $x_2 - x_0 \neq 0$. We obtain

$$\begin{aligned} &\det\left(\begin{bmatrix} x_2 - x_0 & x_2^2 - x_0^2 & x_2^3 - x_0^3 \\ 1 & 2x_1 & 3x_1^2 \\ 0 & 2 & 6x_1 \end{bmatrix}\right) \\ &= (x_2 - x_0)(12x_1^2 - 6x_1^2) - (6x_1(x_2^2 - x_0)^2 - 2(x_2^3 - x_0^3)) \\ &= (x_2 - x_0)(6x_1^2 - 6x_1x_2 - 6x_1x_0 + 2x_2^2 + 2x_2x_0 + 2x_0^2) \\ &= (x_2 - x_0)((c_1 - c_2)^2 + c_1^2 + c_2^2)) \neq 0, \end{aligned}$$

where $c_1 = x_1 - x_0$ and $c_2 = x_2 - x_1$. Since the determinant is nonzero, the matrix is nonsingular, and hence there exists a unique solution.

16. Let us consider the first order ODE

$$\frac{dy}{dt} = f(t, y), \quad t \geq 0,$$
$$y(0) = c, \quad c \in \mathbb{R}.$$

In order to numerically solve this ODE, we introduce a discretisation $t_{n+1} = t_n + h$, where $h > 0$ is the step size, $t_0 = 0$, $n = 1, 2, \ldots$. We compute the numerical approximation for this ODE using the following recurrence formula:

$$y_{n+1} = y_n + \frac{h}{2}\left(f(t_{n+1}, y_n + hf(t_n, y_n)) + f(t_n, y_n)\right), \quad y_0 = c, \quad n = 1, 2, \ldots. \tag{15}$$

Here, y_n is the numerical approximation of the solution y at $t = t_n$, that is $y_n \approx y(t_n)$.

- Suppose that $f(t, y) = \lambda y$, where λ is a complex constant such that $\mathrm{Re}(\lambda) < 0$. Show that for all $n \geq 1$, we have
$$y_n = \rho^n y_0,$$
where $\rho = 1 + h\lambda + \frac{h^2\lambda^2}{2}$. Using this result, derive a condition on the step size h such that the numerical approximation (15) is stable, that is $|y_n| < \infty$ for all $n \geq 0$.
- Write a code to implement (15). Test your code with $f(t, y) = \lambda y$, where $y_0 = 10$ and $\lambda = -1$ is a constant. Run your code until the final time $t = 10$, with various step sizes and compare your results with the analytical solution $y(t) = y_0 e^{\lambda t}$. Try to see what happens if the step size violates the condition that you have derived previously.

Sol:

- Since $f(t, y) = \lambda y$, we have
$$f(t_n, y_n) = \lambda y_n,$$
$$f(t_{n+1}, y_n + hf(t_n, y_n)) = f(t_{n+1}, y_n + h\lambda y_n) = \lambda(y_n + h\lambda y_n) = \lambda y_n + h\lambda^2 y_n.$$
Hence, (15) becomes
$$y_{n+1} = y_n + \frac{h}{2}\left(\lambda y_n + h\lambda^2 y_n + \lambda y_n\right) = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_n.$$
Therefore, for all $n \geq 1$, we have
$$\begin{aligned} y_n &= \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_{n-1} \\ &= \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^2 y_{n-2} = \ldots = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right)^n y_0 = \rho^n y_0, \end{aligned}$$
where $\rho(z) = 1 + z + \frac{z^2}{2}$ and $z = h\lambda \in \mathbb{C}$. To ensure that $|y_n|$ is bounded, ρ must satisfy the condition $|\rho| \leq 1$. For real $\lambda < 0$ this implies $h \leq -2/\lambda$.

See attached Jupyter Notebook for code.

17. Consider the iteration equation

$$\mathbf{u}^{k+1} = A\mathbf{u}^k, \quad \mathbf{u}^0 = \mathbf{f}, \quad k \geq 0, \quad \mathbf{u}^k \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}.$$

What is the necessary condition on the iteration matrix A so that if $\|\mathbf{f}\| < \infty$ then $\|\mathbf{u}^k\| < \infty$ for $k \to \infty$?

Sol: We consider

$$\mathbf{u}^1 = A\mathbf{u}^0 = A\mathbf{f}, \quad \mathbf{u}^2 = A\mathbf{u}^1 = A^2\mathbf{u}^0 = A^2\mathbf{f}, \quad \ldots, \quad \mathbf{u}^k = A^k\mathbf{f},$$

and hence taking the norm of both sides gives

$$\|\mathbf{u}^k\| = \|A^k\mathbf{f}\| \leq \|A^k\|\|\mathbf{f}\|.$$

Note that $\lim_{k\to\infty} \|A^k\| = \infty \iff \rho(A) > 1$, where $\rho(A)$ is the spectral radius of A. Therefore we must have $\rho(A) \leq 1$.