Quiz for Bayesian Inference and Computation, AMSI 2024

Problem 1

Write an `R` function, called `mymax`, which takes input values `(a; b; c)`, and returns the maximum of the three values.

Problem 2

Suppose that $X \sim N(0,1)$, that is, a Gaussian distribution with zero mean and unit variance, and let

$$Y = \frac{\exp(X)}{1+\exp(X)}.$$

Find the probability density function of X and Y.

Problem 3

1. Complete the square of the expression $x^2 + 10x - 3$, i.e., find a, h and k such that

$$x^2 + 10x - 3 = a(x-h)^2 + k.$$

2. Suppose now that $x, b \in \mathcal{R}^p$, $c \in \mathcal{R}$, and that $A \in \mathcal{R}^{p \times p}$ is a symmetric, invertible matrix. Find $h \in R^p$ and $k \in \mathcal{R}$ such that

$$x^\top Ax + x^\top b + c = (x-h)^\top A(x-h) + k.$$

Problem 4

Determine the maximum likelihood estimates of μ and σ^2 when $X_1, X_2, \ldots, X_n$ are independent $N(\mu, \sigma^2)$ variables.

Problem 5

Let $N(x \,|\, \mu_1, \sigma_1^2)$ and $N(x \,|\, \mu_2, \sigma_2^2)$ denote two normal densities. Show that the product of the two densities is proportional to a normal density. What is the expected value and variance of this new *density?*

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Problem 1

```
myMax=function(a,c,b) {
      return(max(a,b,c)
}
```

Problem 2

The density of X is

$$f_X(x) = \exp(-x^2/2)/\sqrt{2\pi}$$

For the density of Y, we use the transformation of random variables theorem

$$f_Y(y) = f_X(h^{-1}(y) \left| \frac{dh^{-1}(y)}{dy} \right|$$

where $y = h(x) = \frac{\exp(x)}{1+\exp(x)}$ and $h^{-1}(y) = \log(y/(1-y))$ and $\frac{d}{dy}h^{-1}(y) = 1/y + 1/(1-y) = 1/(y(1-y))$. Thus

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\log\left(\frac{y}{1-y}\right)^2\right)\frac{1}{y(1-y)}, \quad 0 < y < 1$$

Problem 3

For Part 1, expand

$$a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k$$

then matching the powers of x, we have $a = 1$, $-2ah = 10; h = -5$ and $ah^2 + k = -3; k = -28$.

For Part 2, we know that $A = A^T$, and expanding the right hand side, we have

$$\begin{aligned}(x-h)^T A(x-h) + k &= (x-h)^T(Ax - Ah) + k \\ &= x^T Ax - h^T Ax - x^T Ah + h^T Ah + k\end{aligned}$$

now, since x, h are $p \times 1$, $h^T Ax = x^T Ah$ since this is scalar and

$$h^T Ax = (h^T Ax)^T = x^T A^T h = x^T Ah,$$

so we ave that $b = -2Ah$ and

$$h = -\frac{1}{2}A^{-1}b$$

and $h^T Ah + k = c = \frac{1}{4}(A^{-1}b)^T AA^{-1}b + k$, therefore

$$k = c - \frac{1}{4}(A^{-1}b)^T AA^{-1}b = c - \frac{1}{4}b^T A^{-1}b.$$

Problem 4 The logarithm of the likelihood function is

$$\begin{aligned}\log \mathcal{L}(\mu, \sigma^2) &= \log \prod_{i=1}^{n} f(X_i; \mu, \sigma^2) \\ &= \log \prod_{i=1}^{n} \{(2\pi\sigma^2)^{-1/2} e^{-(X_i-\mu)^2/(2\sigma^2)}\} \\ &= -(n/2)\log(2\pi) - (n/2)\log(\sigma^2) - (2\sigma^2)^{-1} \sum (X_i - \mu)^2\end{aligned}$$

Thus,

$$(\partial/\partial\mu) \log \mathcal{L}(\mu, \sigma^2) = (\sigma^2)^{-1} \sum (X_i - \mu)$$

and

$$(\partial/\partial\sigma^2) \log \mathcal{L}(\mu, \sigma^2) = -(n/2)(\sigma^2)^{-1} + \frac{1}{2}(\sigma^2)^{-2} \sum (X_i - \mu)^2$$

Clearly

$$(\partial/\partial\mu) \log \mathcal{L}(\mu, \sigma^2) = 0 \quad \text{if and only if } \mu = \bar{X}$$

for all $\sigma^2 > 0$. Hence, both partial derivatives equal zero if and only if

$$\mu = \bar{X} \quad \text{and} \quad -(n/2)(\sigma^2)^{-1} + \frac{1}{2}(\sigma^2)^{-2} \sum (X_i - \bar{X})^2 = 0.$$

This is equivalent to

$$\mu = \bar{X} \quad \text{and} \quad \sigma^2 = \frac{1}{n} \sum (X_i - \bar{X})^2.$$

Assuming that this does correspond to a global maximum of $\log \mathcal{L}(\mu, \sigma^2)$ the maximum likelihood estimates of μ and σ^2 are

$$\hat{\mu} = \bar{X} \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \bar{X})^2.$$

Problem 5

A normal density of of the form

$$\begin{aligned} N(x|\mu_n, \sigma_n^2) &\propto \exp\left(-\frac{1}{2\sigma_n^2}(x-\mu_n)^2\right) \\ &\propto \exp\left(-\frac{1}{2\sigma_n^2}x^2 + \frac{\mu_n}{\sigma_n^2}x\right) \qquad (1) \end{aligned}$$

and a product of two normals is

$$\begin{aligned} N(x|\mu_1, \sigma_1^2)N(x|\mu_2, \sigma_2^2) &\propto \exp(-\frac{1}{2\sigma_1^2}(x-\mu_1)^2)\exp(-\frac{1}{2\sigma_2^2}(x-\mu_2)^2) \\ &\propto \exp\left(-\frac{1}{2\sigma_1^2}(x^2 - 2\mu_1 x) - \frac{1}{2\sigma_2^2}(x^2 - 2\mu_2 x)\right) \\ &\propto \exp\left(-\frac{(\sigma_2^2 + \sigma_1^2)}{2\sigma_1^2\sigma_2^2}x^2 + \frac{(\sigma_2^2\mu_1 + \sigma_1^2\mu_2)}{\sigma_1^2\sigma_2^2}x\right) \end{aligned}$$

matching this to the form in (1), we get

$$\frac{1}{\sigma_n^2} = \frac{\sigma_2^2 + \sigma_1^2}{\sigma_1^2\sigma_2^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

and

$$\frac{\mu_n}{\sigma_n^2} = \frac{\sigma_2^2\mu_1 + \sigma_1^2\mu_2}{\sigma_1^2\sigma_2^2} = \frac{1}{\sigma_1^2}\mu_1 + \frac{1}{\sigma_2^2}\mu_2$$

so

$$N(x|\mu_1, \sigma_1^2)N(x|\mu_2, \sigma_2^2) \propto N(x|\mu_n, \sigma_n^2)$$

and

$$\mu_n = w_1\mu_1 + w_2\mu_2, \quad w_1 = \frac{\frac{1}{\sigma_1^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} \text{ and } w_2 = \frac{\frac{1}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

and

$$\sigma_n^2 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}.$$