AMSI Summer School 2024 - A century of harmonic analysis
Preparatory Quiz

The following questions should give you a rough idea of the background required. If you get stuck, try think on it for awhile. If you find that you are still unable to make progress, the solutions can be found below.

1. (a) If $f\colon \mathbb{R}^n \to \mathbb{C}$ is a smooth function, compute the gradient of the function $e^{|f(x)|^2}$.
   (b) For $t > 0$ and $x \in \mathbb{R}^n$, show that the integral
   $$\int_{\mathbb{R}^n} e^{-t|\xi|^2} e^{2ix\cdot\xi} d\xi$$
   converges absolutely, and is equal to $\frac{\pi^{n/2}}{t^{n/2}} e^{-\frac{|x|^2}{t}}$. Throughout the course, $i := \sqrt{-1}$.
   (Hint: First consider the case $t = 1$, and use part (a).)
   (c) For $t > 0$ and $x \in \mathbb{R}^n$, show that the integral
   $$\int_{\mathbb{R}^n} |\xi|^2 e^{-t|\xi|^2} e^{2ix\cdot\xi} d\xi$$
   converges absolutely, and evaluate the integral. (Hint: Use part (b).)

2. Suppose a sequence of functions $\{f_j\colon \mathbb{R} \to \mathbb{R}\}$ converges uniformly to a function $f\colon \mathbb{R} \to \mathbb{R}$, and another sequence of functions $\{g_j\colon \mathbb{R} \to \mathbb{R}\}$ converges uniformly to a function $g\colon \mathbb{R} \to \mathbb{R}$.
   (a) Show that the sum $\{f_j + g_j\}$ converges uniformly to $f + g$ on $\mathbb{R}$.
   (b) Show that the product $\{f_j g_j\}$ might not converge uniformly to fg on $\mathbb{R}$.
   (c) Can you add some assumptions to recover a positive result in part (b)?

3. Let
   $$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$
   where x_i, y_i and a_{ij} are all real numbers.
   (a) Suppose the sum of absolute values of the entries in each row and each column of the square matrix above is at most A, i.e.
   $$\sup_{1\le j\le n} \sum_{i=1}^n |a_{ij}| \le A \quad \text{and} \quad \sup_{1\le i\le n} \sum_{j=1}^n |a_{ij}| \le A.$$
   Show that the Euclidean norm of y is at most A times that of x, i.e.
   $$\Big(\sum_{i=1}^n |y_i|^2\Big)^{1/2} \le A\Big(\sum_{j=1}^n |x_j|^2\Big)^{1/2}.$$
   (b) Under the same assumptions in (a), show that for any $p > 1$,
   $$\Big(\sum_{i=1}^n |y_i|^p\Big)^{1/p} \le A\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}.$$
   (c) Show that if $|a|, |b|, |c| \le 1$, $x_0 = x_{n+1} = 0$ and $x_1, \dots, x_n \in \mathbb{R}$, then
   $$\sum_{i=1}^n |ax_{i-1} + bx_i + cx_{i+1}|^p \le 3^p \sum_{i=1}^n |x_i|^p.$$

4. It would also be nice to try the following multiple choice questions on Terry Tao's webpage:
`http://scherk.pbworks.com/w/page/14864228/Quiz:Inequalities`
`http://scherk.pbworks.com/w/page/14864240/Quiz:Series`
`http://scherk.pbworks.com/w/page/14864230/Quiz3AInnerproductspaces`
(Solutions of these questions are provided on the respective pages.)

Solution:

1. (a) Using the chain rule, we have $\nabla e^{|f(x)|^2} = 2e^{|f(x)|^2}\nabla f(x)$ for every $i = 1, \dots, n$.
   (b) The integral converges absolutely because $\int_{\mathbb{R}^n} e^{-t|\xi|^2} d\xi < \infty$, which in turn follows from the dimension $n = 1$ case using Fubini's theorem.
   For $t = 1$, we verify that
   $$\pi^{-n/2} e^{|x|^2} \int_{\mathbb{R}^n} e^{-|\xi|^2} e^{2ix\cdot\xi} d\xi = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi+ix|^2} d\xi$$
   whereas the last integral is independent of x: in fact, one can differentiate the integral (justify this!) with respect to x, and observe that
   $$\nabla_x \int_{\mathbb{R}^n} e^{-|\xi+ix|^2} d\xi = \int_{\mathbb{R}^n} -2i(\xi + ix) e^{-|\xi+ix|^2} d\xi = \int_{\mathbb{R}^n} i\nabla_\xi e^{-|\xi+ix|^2} d\xi = 0$$
   by the fundamental theorem of calculus. This shows
   $$\pi^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi+ix|^2} d\xi = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|\xi|^2} d\xi = 1,$$
   as desired.
   (c) The integral converges absolutely because $\int_{\mathbb{R}^n} |\xi|^2 e^{-t|\xi|^2} d\xi < \infty$ (the polynomial growth of $|\xi|^2$ at infinity is killed by the exponential decay of $e^{-t|\xi|^2}$ at infinity). In fact, the integral is the $-\frac{\partial}{\partial t}$ derivative of the integral in part (b), which is thus
   $$-\frac{\partial}{\partial t}\Big(\frac{\pi^{n/2}}{t^{n/2}} e^{-\frac{|x|^2}{t}}\Big) = -\frac{\pi^{\frac{n}{2}}}{t^{\frac{n}{2}+2}}\Big(|x|^2 - \frac{n}{2}t\Big) e^{-\frac{|x|^2}{t}}$$
   for $t > 0$ and $x \in \mathbb{R}^n$.

2. (a) Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ so that for all $n \geq N$,
   $$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| \leq \frac{\varepsilon}{2}$$
   It follows that
   $$\begin{aligned}\sup_{x\in\mathbb{R}} |[f_n + g_n](x) - [f+g](x)| &\leq \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| + \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$
   This shows $\{f_n + g_n\}$ converges uniformly to $f + g$ on $\mathbb{R}$.
   (b) Let $f_n(x) = x + \frac{1}{n}$, $g_n(x) = x$, $f(x) = x$ and $g(x) = x$. Then f_n converges to f uniformly on $\mathbb{R}$, and g_n converges uniformly to g on $\mathbb{R}$, but
   $$|f_n(x)g_n(x) - f(x)g(x)| = \Big|x\Big(x + \frac{1}{n}\Big) - x^2\Big| = \frac{|x|}{n},$$
   so
   $$\sup_{x\in\mathbb{R}} |f_n(x)g_n(x) - f(x)g(x)| = \sup_{x\in\mathbb{R}} \frac{|x|}{n} = \infty$$
   for every $n \in \mathbb{N}$, which shows that $\{f_n g_n\}$ does not converge to fg on $\mathbb{R}$.

(c) If additionally both f and g are bounded functions on $\mathbb{R}$, say $\sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |g(x)| \leq M$ for some finite M, then $f_n g_n$ converges uniformly to fg on $\mathbb{R}$. This is because then both $\{f_n\}$ and $\{g_n\}$ are uniformly bounded on $\mathbb{R}$: there exists $M' \in \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}} |f_n(x)| + \sup_{n\in\mathbb{N}} \sup_{x\in\mathbb{R}} |f_n(x)| \leq M'.$$

Then

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |f_n(x)g_n(x) - f(x)g(x)| \\ &\leq \sup_{x\in\mathbb{R}} |f_n(x)||g_n(x) - g(x)| + \sup_{x\in\mathbb{R}} |g(x)||f_n(x) - f(x)| \\ &\leq M' \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| + M \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \to 0 \end{aligned}$$

as $n \to \infty$. This shows $\{f_n g_n\}$ converges uniformly to fg on $\mathbb{R}$ under the additional hypothesis.

3. (a) First, for $i = 1, \dots, n$ we have $y_i = \sum_{j=1}^n a_{ij} x_j$ so

$$|y_i| \leq \sum_{j=1}^n |a_{ij}||x_j| = \sum_{j=1}^n |a_{ij}|^{1/2} (|a_{ij}|^{1/2} |x_j|)$$

By the Cauchy-Schwarz inequality, we have

$$|y_i|^2 \leq \Big(\sum_{j=1}^n |a_{ij}||x_j|^2\Big)\Big(\sum_{j=1}^n |a_{ij}|\Big) \leq A \sum_{j=1}^n |a_{ij}||x_j|^2.$$

Summing over $i = 1, \dots, n$ gives

$$\sum_{i=1}^n |y_i|^2 \leq A \sum_{j=1}^n \Big(\sum_{i=1}^n |a_{ij}|\Big) |x_j|^2 \leq A^2 \sum_{j=1}^n |x_j|^2.$$

This gives the desired inequality.

(b) We can follow a similar proof as above, except that we use Hölder's inequality to

$$|y_i| \leq \sum_{j=1}^n |a_{ij}||x_j| = \sum_{j=1}^n |a_{ij}|^{(p-1)/p} (|a_{ij}|^{1/p} |x_j|)$$

in lieu of Cauchy-Schwarz. We obtain, for every $i = 1, \dots, n$,

$$|y_i|^p \leq \Big(\sum_{j=1}^n |a_{ij}|\Big)^{p-1} \Big(\sum_{j=1}^n |a_{ij}||x_j|^p\Big) \leq A^{p-1} \sum_{j=1}^n |a_{ij}||x_j|^p$$

and hence

$$\sum_{i=1}^n |y_i|^p \leq A^{p-1} \sum_{j=1}^n \Big(\sum_{i=1}^n |a_{ij}|\Big) |x_j|^p \leq A^p \sum_{j=1}^n |x_j|^p.$$

(c) We apply the above result with $a_{ij} = 0$ if $|i - j| > 1$, $a_{ij} = a$ if $j = i - 1$, $a_{ij} = b$ if $j = i$ and $a_{ij} = c$ if $j = i + 1$. We have

$$\sup_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}| \leq 3 \quad \text{and} \quad \sup_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}| \leq 3,$$

and if $y_i = \sum_{j=1}^n a_{ij} x_j$, then

$$y_i = a x_{i-1} + b x_i + c x_{i+1}.$$

So the previous bounds gives

$$\sum_{i=1}^{n} |ax_{i-1} + bx_i + cx_{i+1}|^p \leq 3^p \sum_{j=1}^{n} |x_j|^p.$$