Pre-enrolment Quiz

Lecturer: Volker Schlue (University of Melbourne) **Course title:** General Relativity

Exercise 1. Consider two observers, moving on a line with different, but constant speeds relative to each other. Give a description of this situation in the context of special relativity, and compute the rate of clocks in one system relative to the other.

Exercise 2. State and derive the local conservation law for electric charges in Maxwell's theory.

Exercise 3. Suppose g is a Riemannian metric on the plane, which expressed in polar coordinates (r, φ) takes the form

$$g = \mathrm{d}r^2 + R(r)^2 \mathrm{d}\varphi^2 \,.$$

Compute the arclength of the circles S_r of constant radius r. For which choice of the function $R(r)$ is the geometry of the plane "non-euclidean"?

Exercise 4. A volume form ω on a n-dimensional vector space V is an n-linear form which is totally antisymmetric and nondegenerate. Let V be endowed with an inner product $\langle \cdot, \cdot \rangle$. Show that associated to the inner product $\langle \cdot, \cdot \rangle$ is a volume form ω, which is well-defined by the condition, that for some orthonormal basis $(E_1, \ldots, E_n)$:

$$\omega(E_1, \ldots, E_n) = 1 \,.$$

Exercise 5. Given a finitely extended distribution of masses in Euclidean space, compute the Newtonian gravitational force exerted on a test particle at great distances. In what sense is it true that to "leading order" the force only depends on the total mass of the configuration that creates the gravitational field?

Answer 1. The situation is best described in $1+1$-dimensional Minkowski space. In the reference frame of the first observer we can introduce coordinates (t, x) so that the Minkowski metric takes the form

$$\eta = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2\,. \tag{0.1}$$

The world line of the second observer, moving at speed v relative to the first, is then described by $x = x_0 + vt$. In fact, for any x_0 these lines describe observers which are considered at rest relative to the second frame of reference, and x_0 can be used as a coordinate in that reference frame. Since

$$\mathrm{d}x = \mathrm{d}x_0 + v\mathrm{d}t \tag{0.2}$$

the metric in these coordinates then takes the form

$$\begin{aligned}\eta &= - c^2\mathrm{d}t^2 + (\mathrm{d}x_0 + v\mathrm{d}t)^2 = -(c^2 - v^2)\mathrm{d}t^2 + \mathrm{d}x_0^2 + 2v\mathrm{d}x_0\mathrm{d}t \\ &= - (1 - v^2/c^2)\Big(c\mathrm{d}t - \frac{v/c}{1 - v^2/c^2}\mathrm{d}x_0\Big)^2 + \frac{1}{1 - v^2/c^2}\mathrm{d}x_0^2 \\ &= - \alpha^2\theta^2 + \mathrm{d}\sigma^2\end{aligned} \tag{0.3}$$

Therefore, the rate of clocks measured by the second observer has changed by a factor of

$$\alpha = \sqrt{1 - v^2/c^2} \tag{0.4}$$

in comparison the clocks of the first observer.

Remark 0.1. What does this formula tell you about the measurement of distances?

Answer 2. According to Maxwell's theory, the electric and magnetic fields can be viewed as the components $F_{\mu\nu}$ of the Faraday tensor F, a 2-form on Minkowski space $(\mathbb{R}^{3+1}, \eta)$, satisfying the equations

$$\frac{\partial}{\partial x^\mu}F_{\alpha\beta} + \frac{\partial}{\partial x^\alpha}F_{\beta\mu} + \frac{\partial}{\partial x^\beta}F_{\mu\alpha} = 0 \qquad \frac{\partial}{\partial x^\alpha}F^{\alpha\beta} = J^\beta\,, \tag{0.5}$$

where $x^0 = ct$, and J^μ are the components of the electric current density J, a vectorfield on Minkowski space. Since $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$ is anti-symmetric in $(\mu\nu)$, it follows from the second equation that

$$\frac{\partial}{\partial x^\beta}J^\beta = \frac{\partial^2}{\partial x^\alpha x^\beta}F^{\alpha\beta} = 0\,. \tag{0.6}$$

If we denote by $J^0 = c\rho$ the electric charge density, and by $J^i = \bar{J}^i : i = 1, 2, 3$ the electric current density, in the chosen reference frame, then this becomes the continuity equation,

$$\frac{\partial\rho}{\partial t} + \nabla \cdot \bar{J} = 0\,, \tag{0.7}$$

which expresses in local form the conservation of charge in this theory.

Answer 3. The circles S_r are parametrized by curves $\gamma_r(t) = (r\cos(t), r\sin(t))$ with tangent vector $\dot{\gamma_r}(t) = \frac{\partial}{\partial\theta}$. Hence

$$L[S_r] = \int_0^{2\pi} \sqrt{g(\dot{\gamma_r}(t), \dot{\gamma_r}(t))}\mathrm{d}t = 2\pi R(r)\,. \tag{0.8}$$

Hence the ratio of the circumference of a circle to its diameter is not equal to π unless $R(r) = r$.

Remark 0.2. If r is the distance "as the crow flies" from ANU to any location in Australia, is then $R(r) > r$ or $R(r) < r$ for this geometry to describe the surface of the Earth?

Answer 4. We have to verify that the definition of the volume form ω, by the condition that

$$\omega(E_1, \ldots, E_n) = 1\,, \tag{0.9}$$

does not actually depend on the choice of the orthonormal basis $(E_1, \ldots, E_n)$.

Consider more generally $\omega(X_1, \ldots, X_n)$, were $X_1, \ldots, X_n \in V$. We can expand any vector X_i in the basis $(E_1, \ldots, E_n)$:

$$X_i = \sum_{j=1}^{n} (X_i)^j E_j \quad : i = 1, \ldots, n\,. \tag{0.10}$$

The coefficients $(X_i)^j : j = 1, \ldots, n$ form the i^{th} column of a matrix X. The claim is that by the definition of the determinant,

$$\omega(X_1, \ldots, X_n) = \det(X)\,\omega(E_1, \ldots, E_n)\,. \tag{0.11}$$

Let us verify this directly in the case $n = 2$:

$$X_1 = (X_1)^1 E_1 + (X_1)^2 E_2 \qquad X_2 = (X_2)^1 E_1 + (X_2)^2 E_2\,. \tag{0.12}$$

Then by linearity, and antisymmetry,

$$\begin{aligned}\omega(X_1, X_2) =& (X_1)^1 (X_2)^1 \omega(E_1, E_1) + (X_1)^1 (X_2)^2 \omega(E_1, E_2) \\ &+ (X_1)^2 (X_2)^1 \omega(E_2, E_1) + (X_1)^2 (X_2)^2 \omega(E_2, E_2) \\ =& (X_1)^1 (X_2)^2 - (X_1)^2 (X_2)^1 = \det X\end{aligned} \tag{0.13}$$

Since any two orthonormal bases, of the same orientation, are related by a special orthogonal linear transformation,

$$E'_i = \sum_j O_i{}^j E_j \qquad \det O = 1\,, \tag{0.14}$$

the statement that (0.9) is well-defined follows.

Answer 5. A finitely extended distribution of masses in Euclidean space can be described by a function $\rho : \mathbb{R}^3 \to [0, \infty)$ of compact support which gives the density of mass $\rho(x)$ at a given point $x \in \mathbb{R}^3$. The Newtonian potential $\psi : \mathbb{R}^3 \to \mathbb{R}$ is then determined as the solution of the PDE

$$\Delta\psi = 4\pi\rho \tag{0.15}$$

with the boundary condition that $\psi(x) \to 0 \quad (|x| \to \infty)$. Using the fundamental solution of the Laplacian on $\mathbb{R}^3$, we obtain

$$\psi(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|} \mathrm{d}y\,. \tag{0.16}$$

Since the function ρ is of compact support, we can find $R > 0$ sufficiently large so that $\mathrm{supp}(\rho) \subset B_R(0)$. For distances $|x| \gg R$, we then have that that

$$\psi(x) \simeq -\int_{B_R} \frac{\rho(y)}{|x|} \mathrm{d}y = -\frac{M}{|x|}\,. \tag{0.17}$$

It then follows that the force exerted on a test particle of mass m is

$$F(x) = -m\nabla\psi(x) \simeq -Mm\frac{x}{|x|^3}\,. \tag{0.18}$$

Remark 0.3. In (0.17) we have replaced $|x-y|^{-1}$ for $y \in B_R(0)$ by $|x|^{-1}$ for $|x| \gg R$. Compute the next order in the expansion in $1/|x|$, and characterize its contribution to the force $F(x)$.