Pre-Enrolment Quiz for AMSI Summer School 2024
Course: Algorithmic Game Theory and Economics

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This self-assessment quiz will cover the basics of the following topics:

- Linear algebra.
- Discrete and continuous random variables.
- Single-variable calculus.
- Mathematical analysis.

Problem 1

Consider the following linear system:

$$\begin{aligned} 6x_1 + 4x_2 - x_3 &= y \\ 6x_3 - 2x_1 - x_2 &= y \\ 2x_1 + 3x_2 + x_3 &= y \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

(a) Rewrite the linear system in the form of $\mathbf{A} \cdot \mathbf{v} = \mathbf{b}$, where $\mathbf{A}$ is a 4×4 matrix, $\mathbf{v} = [x_1, x_2, x_3, y]^\mathsf{T}$, and $\mathbf{b}$ is a vector whose entries do not depend on x_1, x_2, x_3, y.

(b) What is the determinant of the matrix $\mathbf{A}$? Is $\mathbf{A}$ invertible?

(c) Solve the linear system.

Problem 2

A random variable X has the following probability density function f, wherein c is a positive real number:

$$f(x) = \begin{cases} c \cdot (3 - |2x - 4|)\,, & \text{for } 0.5 \leq x \leq 3.5 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Plot f.

(b) Explain why $c = 2/9$.

(c) What are the expected value and the variance of the random variable X?

(d) Recall that $F(y) = \mathbb{P}[X \geq y]$ is the cumulative distribution function of the random variable X.

  (i) Write down $F(y)$ explicitly for $2 \leq y \leq 3.5$.

  (ii) Write down $F(y)$ explicitly for $0.5 \leq y \leq 2$.

(e) What is the value of y that maximizes $y \cdot F(y)$?

Problem 3

Let $u : (\mathbb{R}_+)^3 \to \mathbb{R}_+$ be the following function:

$$u(x, y, z) \;=\; x^{1/2} + 2y^{1/2} + 3z^{1/2}.$$

Subject to the constraint $4x + 2y + z \leq 10$, compute (x, y, z) which maximize the value of u. (Hint: use the method of Lagrangian multipliers.)

Problem 4

A gradient descent process on a single-variable smooth function f is specified by an initial point x_0, a step-size $\lambda > 0$, and the following update rule for $n \geq 1$:

$$x_n \;=\; x_{n-1} - \lambda \cdot f'(x_{n-1}).$$

Suppose $f(x) = (x-3)^4$; clearly, the minimum of f is attained at $x = 3$. Suppose the initial point of the gradient descent process is $x_0 = 5$.

(a) Write down the gradient descent update rule explicitly.

(b) If $\lambda > 1/8$, explain why the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the gradient descent process does not converge to 3. (Hint: show that $|x_n - 3|$ strictly increases for all $n \geq 0$.)

(c) For any $0 < \lambda < 1/8$, show that $\lim_{n \to \infty} x_n = 3$. In other words, the gradient descent process does converge to the minimum point of f.

Problem 5

Recall that a function $h : \mathbb{R}^n \to \mathbb{R}$ is convex if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,

$$h\left(\lambda \cdot \mathbf{x} + (1-\lambda) \cdot \mathbf{y}\right) \;\leq\; \lambda \cdot h(\mathbf{x}) + (1-\lambda) \cdot h(\mathbf{y}).$$

(a) Suppose f_1, f_2 are convex functions. Prove that $f_1 + f_2$ is also a convex function.

(b) Let $g : \mathbb{R}_+^n \to \mathbb{R}$ be the function defined by:

$$g(x_1, x_2, \ldots, x_n) \;=\; -\log(x_1 + x_2 + \ldots + x_n).$$

Prove that g is a convex function. (Hint: you may use the Power-Mean inequality.)

Solution

Problem 1

(a) The linear system can be rewritten as

$$\begin{bmatrix} 6 & 4 & -1 & -1 \\ -2 & -1 & 6 & -1 \\ 2 & 3 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Use Gaussian elimination to bring the matrix to a upper-triangular form, then the determinant is the product of the diagonal entries. Below shows several intermediate matrices. The determinant of $\mathbf{A}$ is $6 \times \frac{1}{3} \times (-27) \times \frac{1}{2} = -27$. $\mathbf{A}$ is invertible.

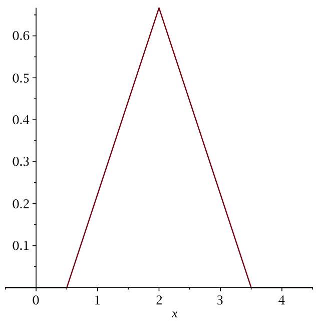
$$\begin{bmatrix} 6 & 4 & -1 & -1 \\ -2 & -1 & 6 & -1 \\ 2 & 3 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & -1 & -1 \\ 0 & \frac{1}{3} & \frac{17}{3} & -\frac{4}{3} \\ 2 & 3 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & -1 & -1 \\ 0 & \frac{1}{3} & \frac{17}{3} & -\frac{4}{3} \\ 0 & 0 & -27 & 6 \\ 1 & 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & -1 & -1 \\ 0 & \frac{1}{3} & \frac{17}{3} & -\frac{4}{3} \\ 0 & 0 & -27 & 6 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(c) We can solve the linear system using Gaussian elimination. The solution is $x_1 = 1/9$, $x_2 = 4/9$, $x_3 = 4/9$ and $y = 2$.

Game/Econ Teaser: For the normal-form two-person zero-sum game represented by the matrix $\begin{bmatrix} 6 & 4 & -1 \\ -2 & -1 & 6 \\ 2 & 3 & 1 \end{bmatrix}$, the mixed strategy of one of the players at the Nash equilibrium is $(x_1, x_2, x_3) = (\frac{1}{9}, \frac{4}{9}, \frac{4}{9})$.

Problem 2

(a)



(b) Since f is a probability density function, $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1$. The integral is same as the area of the triangle plotted in part (a), thus $\frac{(3c)\cdot 3}{2} = 1$, and hence $c = 2/9$.

(c) f is symmetric across $x = 2$, so $\mathbb{E}[X] = 2$. The variance of X is given by $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[X^2] - 4$. We compute $\mathbb{E}[X^2]$ as below:

$$\int_{0.5}^{2} x^2 \cdot \frac{2}{9} \cdot (2x-1)\,\mathrm{d}x + \int_{2}^{3.5} x^2 \cdot \frac{2}{9} \cdot (7-2x)\,\mathrm{d}x = \frac{35}{8}.$$

Thus the variance of X is $3/8$.

(d) (i) $\frac{49}{18} - \frac{14}{9}y + \frac{2}{9}y^2$.
(ii) $\frac{17}{18} + \frac{2}{9}y - \frac{2}{9}y^2$.

(e) Using part (d), it suffices to find out the maximum of $y\left(\frac{17}{18}+\frac{2}{9}y-\frac{2}{9}y^2\right)$ for $0.5 \leq y \leq 2$, and the maximum of $y\left(\frac{49}{18}-\frac{14}{9}y+\frac{2}{9}y^2\right)$ for $2 \leq y \leq 3.5$.

For $0.5 \leq y \leq 2$, $\frac{\mathsf{d}}{\mathsf{d}y}\left[y\left(\frac{17}{18}+\frac{2}{9}y-\frac{2}{9}y^2\right)\right] = -\frac{2}{3}y^2+\frac{4}{9}y+\frac{17}{18}$. This quadratic equation has root $\frac{1}{3}+\frac{\sqrt{55}}{6}$ which is between 0.5 and 2. It is easy to verify that this is the maximum point of $y \cdot F(y)$ in the interval $0.5 \leq y \leq 2$.

For $2 \leq y \leq 3.5$, it is not hard to verify that $y \cdot F(y)$ is decreasing in the interval, so the maximum of $y \cdot F(y)$ in this interval is attained at $y = 2$.

Overall, the global maximum of $y \cdot F(y)$ is attained at $y = \frac{1}{3}+\frac{\sqrt{55}}{6} \approx 1.5694$.

Game/Econ Teaser: If a seller sells an item to a bidder whose valuation follows the above probability distribution, for the seller to optimize expected revenue while avoiding strategic manipulation from the bidder, the seller should set the price of the item to $\frac{1}{3}+\frac{\sqrt{55}}{6}$.

Problem 3

The Lagrangian dual is $\mathcal{L}(x,y,z;\lambda) = x^{1/2}+2y^{1/2}+3z^{1/2}+\lambda(10-4x-2y-z)$. The stationary point occurs at $\frac{\partial\mathcal{L}}{\partial x} = \frac{\partial\mathcal{L}}{\partial y} = \frac{\partial\mathcal{L}}{\partial z} = \frac{\partial\mathcal{L}}{\partial \lambda} = 0$, i.e.,

$$\frac{1}{2}x^{-1/2}-4\lambda \;=\; y^{-1/2}-2\lambda \;=\; \frac{3}{2}z^{-1/2}-\lambda \;=\; 10-4x-2y-z \;=\; 0.$$

We then have $x = (8\lambda)^{-2}$, $y = (2\lambda)^{-2}$ and $z = (2\lambda/3)^{-2}$. Since $4x+2y+z = 10$, we have

$$\left(4\cdot\frac{1}{64}+2\cdot\frac{1}{4}+\frac{9}{4}\right)\lambda^{-2} \;=\; 10,$$

and hence $\lambda^{-2} = 32/9$. Hence, $x = \frac{1}{64}\cdot\frac{32}{9} = \frac{1}{18}$, $y = \frac{1}{4}\cdot\frac{32}{9} = \frac{8}{9}$, and $z = \frac{9}{4}\cdot\frac{32}{9} = 8$.

Game/Econ Teaser: Suppose there are three goods. Suppose a buyer's utility function is u, and her budget is \$10. If the unit-prices of the three goods are \$4, \$2 and \$1 respectively, then the buyer's demand is $\frac{1}{18}$ unit of the first good, $\frac{8}{9}$ unit of the second good, and 8 units of the third good.

Problem 4

1. $x_n = x_{n-1} - 4\lambda(x_{n-1}-3)^3$.
2. Suppose $x_{n-1} = 3+a$. Then $x_n - 3 = a - 4\lambda a^3 = a(1-4\lambda a^2)$. Observe that if $|a| \geq 2$ and $\lambda > 1/8$, then $|1-4\lambda a^2| > 1$, thus $|x_n - 3| > |x_{n-1}-3|$. By using the above observation, and by induction, we can show that when $x_0 = 5$, $|x_n - 3|$ strictly increases for all $n \geq 0$.
3. Observe that if $|a| \leq 2$ and $0 < \lambda < 1/8$, then $|1-4\lambda a^2| < 1$, thus $|x_n-3| \leq (1-\epsilon)\cdot|x_{n-1}-3|$ for some small ϵ. By using the above observation, and by induction, we can see that when $x_0 = 5$, $\lim_{n\to\infty} x_n = 3$.

Game/Econ Teaser: In the summer course, you will learn about the intimate connections between economies and optimisation. You will see that some dynamics in economic systems are equivalent to gradient or mirror descents. The above example shows that too aggressive descent updates (i.e., updates with too large step-sizes) can fail to converge to equilibrium.

Problem 5

(a) Let $f_3 = f_1 + f_2$.

$$\begin{aligned} f_3(\lambda\cdot\mathbf{x}+(1-\lambda)\cdot\mathbf{y}) =\; & f_1(\lambda\cdot\mathbf{x}+(1-\lambda)\cdot\mathbf{y}) + f_2(\lambda\cdot\mathbf{x}+(1-\lambda)\cdot\mathbf{y}) \\ \leq\; & \lambda\cdot f_1(\mathbf{x})+(1-\lambda)\cdot f_1(\mathbf{y})+\lambda\cdot f_2(\mathbf{x})+(1-\lambda)\cdot f_2(\mathbf{y}) \\ & \qquad\qquad\qquad \text{(due to convexity of } f_1 \text{ and } f_2\text{)} \\ =\; & \lambda\cdot f_3(\mathbf{x})+(1-\lambda)\cdot f_3(\mathbf{y}) \quad \text{(by definition of } f_3\text{)}. \end{aligned}$$

This shows f_3 is a convex function.

(b) Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Let $X = x_1 + x_2 + \ldots + x_n$ and $Y = y_1 + y_2 + \ldots + y_n$. Then

$$g(\lambda \cdot \mathbf{x} + (1-\lambda) \cdot \mathbf{y}) = -\log(\lambda X + (1-\lambda)Y),$$

and

$$\lambda \cdot g(\mathbf{x}) + (1-\lambda) \cdot \mathbf{y} = -\lambda \log X - (1-\lambda) \log Y = -\log(X^{\lambda} Y^{1-\lambda}).$$

To show that g is convex, it suffices to show that $\lambda X + (1-\lambda)Y \geq X^{\lambda} Y^{1-\lambda}$, which holds due to the Power-Mean inequality.

Game/Econ Teaser: In optimisation, machine learning and their connections to economies, convexity is a foundational concept.